AN ATTACK ON THE KERVAIRE INVARIANT CONJECTURE

by Michael Weiss

Thesis submitted for the degree of Doctor of Philosophy .

Warwick University,
Mathematics Department,
Coventry CV 4 7 AL

January, 1982 .

TABLE OF CONTENTS

0. Introduction

I. The generating problem

II. Algebraic preliminaries

III. The groups $S_{2k}(\cdots)$

IV. Proofs

V. More proofs

VI. Examples and applications

References

ACKNOWLEDGMENTS

I wish to thank my supervisor, Brian Sanderson, for his guidance in general and for suggesting the Kervaire invariant conjecture as a research topic (for that is what this thesis was originally supposed to be about). Further, I am indebted to David Epstein for some encouragement and advice, and to the Mathematics Institute staff as a whole for providing an atmosphere which, to any German student, must appear singularly un-German .

SUMMARY

The present thesis developed in three stages, each associated with a particular problem. The problems are, in 'historical' order:

(1) (The Kervaire invariant problem): Find out for which positive integers k the Kervaire invariant $\Phi : \pi^s_{2k} \longrightarrow Z_2$ is onto.

(2) Given a fibration $f: B \longrightarrow BO$, with associated Thom spectrum $M(B,f)$ [STONG], is there any useful connection between the surgery obstruction group $L_{2k}(\pi_1(B), \pi_1(f))$ on one hand and the bordism group $\pi_{2k}(M(B,f))$ on the other ?

(3) Construct a general theory of signature invariants which will please the handlebody theorist.

Of course (2) and (3) are just means of escaping from (1).

Ad (3): The handlebody theorist under consideration is meant to ask questions such as: "Given a class x in the bordism group $\pi_{2k}(M(B,f))$, can I find a (B,f)-manifold N^{2k} representing x, with a handle decomposition having no handles of index k ?" To solve problems of this type, signature groups $S_{2k}(B,f)$ and obstruction homomorphisms $\sigma: \pi_{2k}(M(B,f)) \longrightarrow S_{2k}(B,f)$ are designed (both functorial). The construction of the groups $S_{2k}(B,f)$ is in two steps. First, certain algebraic data have to be extracted from the fibration $f: B \longrightarrow BO$ – the main thing here is a more pompous version of the 'total Wu class' (in $H^*(B;Z_2)$) associated with f. Second, the group $S_{2k}(B,f)$ can be described (and sometimes computed) in terms of these data, as a variation on the Witt-Grothendieck theme. (I must admit that the description I offer is very clumsy, though it seems to describe the right thing .)

Ad (2): There is a useful connection, in the shape of a natural homomorphism $\lambda: L_{2k}(\pi_1(B), \pi_1(f)) \longrightarrow S_{2k}(B,f)$.

Ad (1): It is suggested, and supported by various examples, that the signature theory just sketched might help disprove (but not prove) the Kervaire Invariant Conjecture (which postulates that Φ is onto precisely if $k+1$ is a power of 2).

Elements of $S_{2k}(B,f)$, or at least some elements, can be considered as obstructions to solving a certain geometric problem, which is described in the first half of section I (this is still more or less introductory). The remainder of the section outlines the ideas used in translating the geometric problem into an algebraic one.

The translation process falls into two parts: first, extraction of an algebraic invariant from the given fibration $f:B \longrightarrow BO$, or from the resulting stable vector bundle over B – I now tend to think of this invariant as a 'generalized total Wu class' (cf. VI.3); and second, construction of the groups $S_{2k}(B,f)$ in terms of the invariant . These are treated, in reverse order, in sections II and III; II also contains the definition of $\lambda: L_{2k}(\pi_1(B), \pi_1(f)) \longrightarrow S_{2k}(B,f)$, and III that of $\delta: \pi_{2k}(M(B,f)) \longrightarrow S_{2k}(B,f)$.

Technical proofs can be found in sections IV and V ; on the whole, in IV if they involve geometry, in V otherwise.

Section VI , the final one, consists of examples and applications, as it ought to . They deal mostly with the Kervaire invariant problem (viz: "For which positive integers n is the Kervaire-Arf invariant

$$\Phi : \pi^s_{2n} \longrightarrow Z_2$$

nontrivial ? ") . For a survey, see [JONES-REES] .
The first application is a weird proof of Browder's Theorem ([BROWDER]) stating that Φ is trivial except possibly when $n = 2^p - 1$ for some integer $p \geqslant 1$. (It is known to be, or rumoured to be, nontrivial for $n = 2^p - 1$, $p < 7$.) This is based on a computation of $S_{2k}(B,f)$ for the case $B = \mathbb{R}P^\infty$, $f = * =$ constant map . (The associated

bordism theory is that of 'framed manifolds with double covers'.)
By [KAHN-PRIDDY], the transfer homomorphism (obtained by 'concentrating' on the double cover)

$$\pi_n(M(\mathbb{R}P^\infty, *)) \longrightarrow \pi_n^s$$

is surjective for $n>0$.
A transfer also exists in the S-theory at issue; it gives a commutative square (k odd)

$$\begin{array}{ccc}
\pi_{2k}(M(\mathbb{R}P^\infty, *)) & \xrightarrow{\text{transfer}} & \pi^s_{2k} \\
\big\downarrow \sigma & & \big\downarrow \sigma \\
S_{2k}(\mathbb{R}P^\infty, *) & \xrightarrow[\text{transfer}]{} & S_{2k}(*) \cong L_{2k}(\{1\}) \\
 & & \cong \\
 & & Z_2
\end{array}$$

This shows that the Kervaire-Arf invariant Φ can be nontrivial in dimension $2k$ only if the transfer from $S_{2k}(\mathbb{R}P^\infty, *)$ to $S_{2k}(*)$ is nontrivial; but it so happens that the said transfer is nontrivial precisely if $k = 2^p-1$ for some integer $p \geq 1$ (given that k is odd), which concludes the proof.

Admittedly, the main thrust here comes from the Kahn-Priddy theorem. Still, it is a useful illustration of the theory, and suggests that more thorough investigations might give more restrictive conditions (for nontriviality of Φ).

The second item explains what Wu classes have to do with the theory and shows that the theory of [BROWN 1], when applied to Thom spectra, is the same as 'S-theory with coefficients Z_2'.

Finally, and somewhat out of the blue, another attack on the Kervaire invariant problem involving the Brown-Gitler spectrum $B(k)$ [BROWN-GITLER] is suggested. It is based on the following argument.

Suppose $M(B,f)$ is a Thom spectrum (arising from a fibration $f:B \longrightarrow BO$) such that the map $j:Z_2 \longrightarrow S_{2k}(B,f)$ defined by commutativity of the diagram

$$\begin{tikzcd} Z_2 \arrow[r, phantom, "\cong"] \arrow[rrr, bend right, "j"'] & L_{2k}(\{1\}) \arrow[r, "\lambda", "\cong"'] & S_{2k}(P(BO),e) \arrow[r, "i"] & S_{2k}(B,f) \end{tikzcd}$$

is injective ($e: P(BO) \longrightarrow BO$ is the path fibration, and i is induced by a suitable map over BO ; also, $k = 2^p-1$).
Incidentally, this will be the case if $M(B,f)$ is a Wu-k-spectrum in the sense of [BROWN 1] .
Then, from the commutative diagram

$$\begin{tikzcd} \pi_{2k}(S^0) \cong \pi_{2k}(M(P(BO)\ e)) \arrow[r, "b"] \arrow[d] & S_{2k}(P(BO),e)) \arrow[d, "i"] & L_{2k}(\{1\}) \cong Z_2 \arrow[l, "\lambda"'] \arrow[dl, "j"] \arrow[d] \\ \pi_{2k}(M(B,f)) \arrow[r, "b"] & S_{2k}(B,f) & L_{2k}(\pi_1(B), \pi_1(f)) \arrow[l, "\lambda"'] \end{tikzcd}$$,

a valuable necessary condition can be extracted:
$z \in \pi_{2k}(M(B,f))$ can be the image of a Kervaire-invariant-1 element in $\pi_{2k}(S^0)$ only if $b(z) = j(1) \in S_{2k}(B,f)$. Together with other 'ad hoc' conditions, this might just be sufficient to show that no element in $\pi_{2k}(M(B,f))$ can possibly be the image of a Kervaire-invariant-1 element in π^s_{2k} – hence, that there are no such elements in π^s_{2k} .

This strategy can be carried out to some extent if $M(B,f)$ is a Brown-Gitler spectrum .

I. THE GENERATING PROBLEM

The groups $S_{2k}(\ldots)$ are obstruction groups in a certain sense, obstructing the existence of a solution to the following problem:

I.1 PROBLEM: Given $x \in \pi_{2k}(M(B,f))$, can we find a (B,f)-manifold N^{2k} representing the 'bordism class' x, and a handle decomposition for N having no k-handles ? (The handle decomposition will also be required to have 'semi-torsion 0' ; see the definition I.2 below.)

The point is this: Suppose we have two elements $y,z \in \pi_{2k}(M(B,f))$. Say that y is k-equivalent to z if representing manifolds N_y^{2k}, N_z^{2k} exist, with handle decompositions H_y, H_z respectively, such that the 'k-skeleton of N_y' is diffeomorphic (as a (B,f)-manifold) to the 'k-skeleton of N_z'. (Explanation: k-skeleton = union of j-handles for $j \leq k$.)

Observation (i): y is k-equivalent to z if and only if problem I.1 above (with the semi-torsion clause deleted) can be solved for $x = y-z$. (Hence k-equivalence is indeed an equivalence relation.)

Observation (ii): Suppose $\varphi: \pi_{2k}(M(B,f)) \longrightarrow G$ is any homomorphism (G any abelian group). If φ wants to be called a signature invariant, and if $y \in \pi_{2k}(M(B,f))$ is k-equivalent to $z \in \pi_{2k}(M(B,f))$, then we (almost) have a right to expect: $\varphi(y) = \varphi(z)$.

I.2 DEFINITION (of semi-torsion):

Suppose N^{2k} has a handle decomposition with no k-handles. This gives us a simple homotopy equivalence $N \longrightarrow X$, where X is a finite

CW-complex (with one j-cell for each j-handle) . Let $C(\tilde{X})$ be the cellular chain complex of the universal covering space of X ; this is a free chain complex over $\mathbb{Z}[\pi_1(X)]$.

Transporting Poincaré duality from N to X , we obtain a simple homotopy equivalence $\phi : C(\tilde{X})^* \longrightarrow C(\tilde{X})$ of degree 2k (cap product with the fundamental class).

Now X has no k-cells; so $C(\tilde{X})$ and $C(\tilde{X})^*$ are trivial in dimension k , and ϕ splits into two 'partial' homotopy equivalences (below and above dimension k). Let $t \in Wh(\pi_1(X))$ be the torsion of the 'bottom half of ϕ ' ; then the image of t in $Wh(\pi_1(B))$ is what I call the semi-torsion (of the given handle decomposition without k-handles).

I.3 EXAMPLE: Let $x \in \pi_{2k}(M(B,f))$ be a bordism class which does have a representative N^{2k} as above, i.e. one with a handle decomposition having no k-handles, and semi-torsion 0 . If B has only finitely many components, each with a finitely presented fundamental group, then we can arrange the maps $\pi_0(N) \longrightarrow \pi_0(B)$ and $\pi_1(N) \longrightarrow \pi_1(B)$ (induced by the classifying map for the stable normal bundle of N) to be isomorphisms . (If $k > 2$, this can be done without spoiling the given handle decomposition in any way, by performing surgery on 0- and 1-spheres.)

Put N_- = (k-1)-skeleton of N = union of j-handles for $j < k$, and $N_+ = N - \mathrm{Int}(N_-)$. N_- can be written as $P^{2k-1} \times I$ (where P^{2k-1} is a (B,f)-manifold with boundary). Then, of course, the inclusion $P \times \{0\} \hookrightarrow N_-$ is a simple homotopy equivalence. But we also have an inclusion $P \times \{0\} \hookrightarrow N_+$, since $P \times \{0\} \subset \partial(N_-) = \partial(N_+)$; computations with homology show that this, too, is a homotopy equivalence. The vanishing of the semi-torsion now implies that it is in fact a simple homotopy equivalence. This shows that N^{2k} has an open book decomposition with page P (see [QUINN]).

(The page here is rather special. Quinn shows that if the closed manifold N^{2k} admits an open book decomposition at all, it admits one with a canonical page — i.e. a page which possesses a handle decomposition having only j-handles for $j \leq k$. However, P above admits a handle decomposition having only j-handles for $j < k$.)

I.3 A GENERALIZATION OF PROBLEM I.1 (see also I.5 below):

Each $K \in L_{2k}(\pi_1(B), \pi_1(f))$ gives rise to a variant of I.1 above, as follows.

Given $z \in \pi_{2k}(M(B,f))$ (as well as $K \in L_{2k}(\pi_1(B), \pi_1(f))$), we want to know: Does there exist a (B,f)-surgery problem*)

$$\begin{tikzcd} N^{2k} \arrow[dr, "\gamma"] \arrow[d] & \\ X \arrow[r, "g"'] & B \end{tikzcd}$$

such that (i) X is a (finite, simple) Poincaré duality CW-space with no k-cells and semi-torsion 0 ;

(ii) N represents the bordism class z ;

(iii) the surgery obstruction (in $L_{2k}(\pi_1(X), \pi_1(f \cdot g))$) has image K in $L_{2k}(\pi_1(B), \pi_1(f))$

?

*) A surgery problem in which the stable bundle over X is classified by a map $X \xrightarrow[g]{} B$.

I.3 is only a reformulation of I.1 if we put $\kappa = 0$ (and if $k>2$, say).

I.4 PROMISE: Let $K \subset \pi_{2k}(M(B,f)) \oplus L_{2k}(\pi_1(B), \pi_1(f))$ be the subgroup consisting of all those (z, κ) for which problem I.3 has a solution. We may view K as the graph of a homomorphism from a subgroup of $\pi_{2k}(M(B;f))$ (namely the image of K under the projection onto the first factor) to a factor group of $L_{2k}(\pi_1(B), \pi_1(f))$ (namely $L_{2k}(\cdots)/\,K \cap L_{2k}(\cdots)$). This homomorphism $\Psi_{B,f}$ is natural with respect to maps over BO. The functor S_{2k} and the natural transformations σ, λ mentioned in section 0 are intended as tools for studying $\Psi_{B,f}$:

$$K \subset \pi_{2k}(M(B,f)) \oplus L_{2k}(\pi_1(B), \pi_1(f))$$

is precisely the kernel of the homomorphism

$$\pi_{2k}(M(B,f)) \oplus L_{2k}(\pi_1(B), \pi_1(f)) \xrightarrow{\ \sigma \oplus -\lambda\ } S_{2k}(B,f) .$$

I.5 APOLOGY:

The formulation given in I.3 is reckless, because basepoints and orientations have been ignored. Recall that the surgery obstruction group $L_n(\pi, w)$ has a geometric description (if $n \geqslant 5$) as the bordism group of 'surgery problems with boundary over (π, w), solved on the boundary'. Such a surgery problem etc. consists of

- a finite Poincare pair (Y,X) and bundle γ over Y, compact manifold N with boundary M, with $\dim(N) = n$;
- a map $\phi : (N,M) \longrightarrow (Y,X)$ of pairs, inducing a simple homotopy equivalence $M \simeq X$, and satisfying $\phi^*(w_Y) = w_N$ (where w_Y, w_N are the first Stiefel-Whitney classes);

- a stable framing of $\tau_N \oplus \phi^*\gamma$;

- a principal π-bundle over Y , say $p: \tilde{Y} \longrightarrow Y$, inducing a principal π-bundle over N , say $\phi^* p: \tilde{N} \longrightarrow N$;

- an isomorphism of the twofold cover $\phi^* p \otimes w : \tilde{N}_w \longrightarrow N$ (explained below) with the orientation cover of N (in particular, the compact manifold $\tilde{N}_w$ is then oriented).

Finally, ϕ is required to have 'degree 1'; that is, the homomorphism

$$H_n(\tilde{N}_w, \tilde{M}_w; Z) \longrightarrow H_n(\tilde{Y}_w, \tilde{X}_w; Z)$$

should send <u>the</u> fundamental class in the former group to <u>a</u> fundamental class in the latter group .

EXPLANATION: At a point $x \in N$, the bundle $\phi^* p \otimes w$ has fibre

$$[\phi^* p \otimes w]^{-1}(x) \quad := \quad \text{set of } \pi\text{-orbits of} \quad [\phi^* p]^{-1}(x) \times \{0,1\}$$

(here π acts on the left factor anyway, and on the right factor $\{0,1\}$ via $\pi \xrightarrow{w} Z_2$ = permutations of $\{0,1\}$). A similar construction gives $p \otimes w: \tilde{Y}_w \longrightarrow Y$.

Returning to B (and $f: B \longrightarrow BO$) , we find that there is one group $L_{2k}(\pi_1(B;x),w)$ for every point $x \in B$, and hence a locally constant sheaf of groups over B . This sheaf may be nontrivial, even if B is connected (which we will assume for simplicity) ; for a fixed $x \in B$, $\pi_1(B,x)$ acts on the stalk $L_{2k}(\pi_1(B;x),w)$ by the rule

$$\alpha \cdot K = (-1)^{w(\alpha)} \cdot K \qquad (\alpha \in \pi_1 \, , \, K \in L_{2k}).$$

However, the induced sheaf over the orientation cover B^{or} is trivial, i.e. globally constant, and may therefore be denoted by $L_{2k}(\pi_1(B),w)$.

(A point in B^{or} is the same as a point in B plus an orientation of the 'stable' vector space attached to that point.)

A proper reformulation of I.3 (left to the reader) gives a definition of K (from I.4) as a (globally constant) subsheaf of the (globally constant) sheaf $\pi_{2k}(M(B,f)) \oplus L_{2k}(\pi_1(B),w)$ over B^{or}.

Finally, a dimension restriction is of course necessary in I.4: $k > 2$.

How can problems like I.1 or I.3 be translated into algebraic problems? The central ideas employed in the present paper are given below, in unpolished form. Proposition I.7 is particularly important.

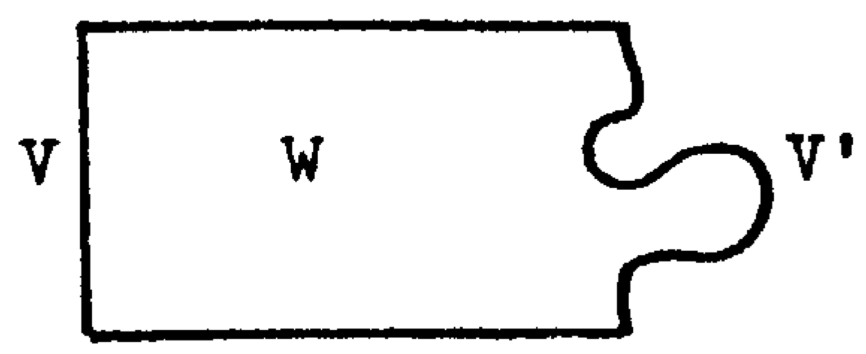
I.6 PREPARATIONS:

(i) Let $f: B \longrightarrow BO$ be given as usual, and suppose e.g. that B has the homotopy type of a CW-complex with finitely many cells in each dimension. Fix $k>2$, and choose a compact (B,f)-manifold V^{2k-1} (with boundary) such that

- V has a handle decomposition with handles of index $< k$ only;
- the classifying map for the normal bundle ν_V, which goes from V to B, is a $(k-1)$-equivalence [SWITZER 3.17].

Such a V certainly exists; it is a 'thickened' $(k-1)$-skeleton for B (among other things), made up of handles instead of cells.

(ii) Let $\mathcal{S}(V)$ be the set of equivalence classes of '(B,f)-cobordisms modulo boundary from V to something else'. Thus every element of $\mathcal{S}(V)$ is represented by a (B,f)-cobordism $(W^{2k}; V^{2k-1}, V'^{2k-1})$ (or rather a 'triad' of (B,f)-manifolds, in the notation of [MILNOR 1]) which is a cobordism modulo boundary (picture):



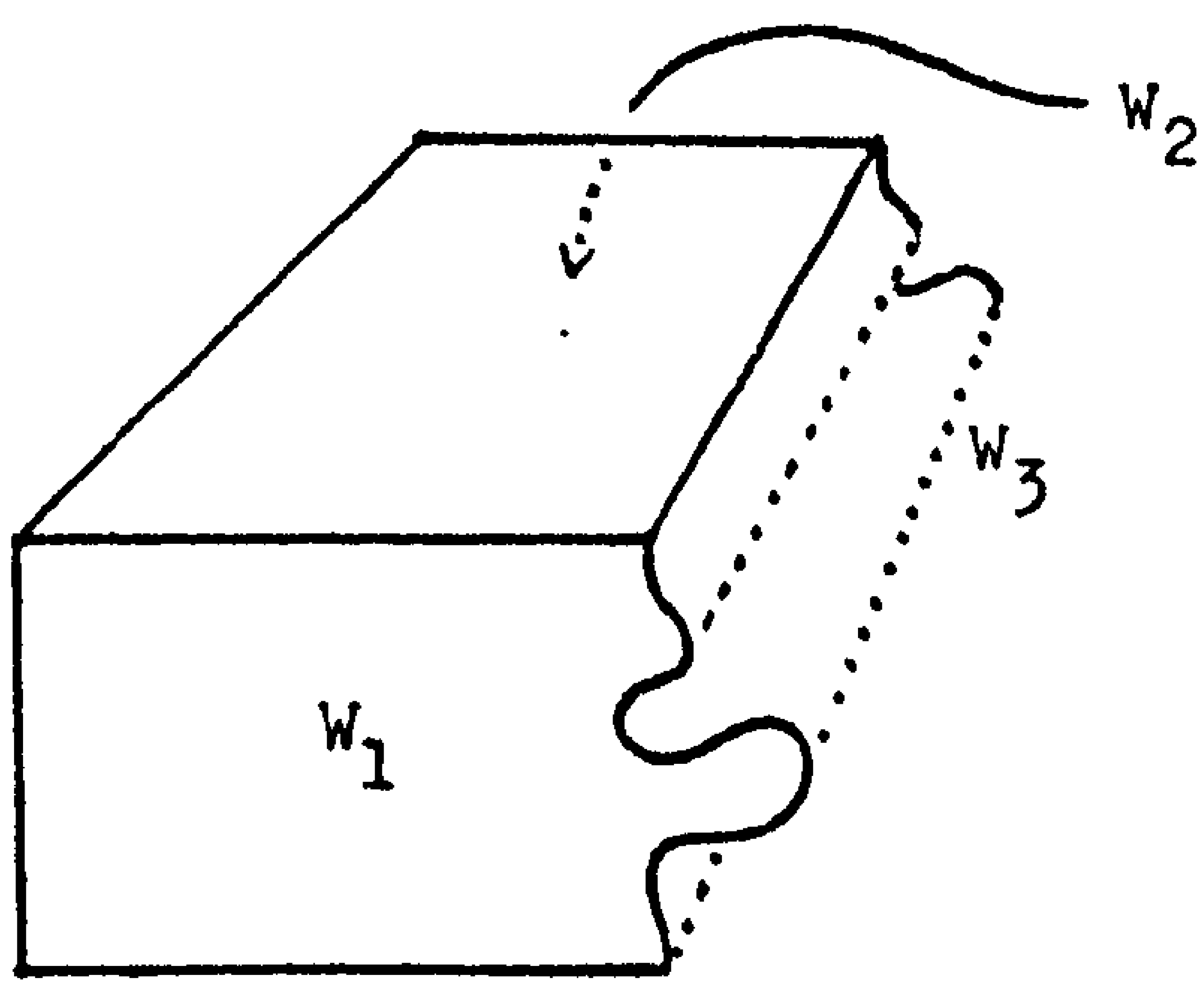
;

two such cobordisms $(W_1;V,V')$ and $(W_2;V,V'')$ are equivalent if (roughly) W_1 and W_2 are (B,f)-cobordant modulo boundary.

To be more precise, $(W_1;V,V')$ is equivalent to $(W_2;V,V'')$ iff there exists an s-cobordism $(W_3;V',V'')$ – also a (B,f)-cobordism modulo boundary (which makes sense because V' and V'' do have the same boundary) – such that the closed (B,f)-manifold

$$Z := W_1 \cup (V \times I) \cup ((\partial V \times I) \times I) \cup W_3 \cup -W_2$$

represents $0 \in \pi_{2k}(M(B,f))$. Picture of Z :



(iii) Define a map (of sets)

$$\partial_V : \pi_{2k}(M(B,f)) \longrightarrow \mathcal{S}(V)$$

by mapping the bordism class represented by N^{2k}, say, to the equivalence class of $(V \times I \cup N;\ V \cong V\times\{0\},\ V\times\{1\})$; that is , by adding to the interior of the trivial cobordism.

(iv) Define a map $\lambda_V: L_{2k}(\pi_1(B), \pi_1(f)) \longrightarrow \mathcal{S}(V)$ as follows.

According to [WALL 1, Thm.5.8] there exists, for any $\kappa \in L_{2k}(\pi_1(B), \pi_1(f)) = L_{2k}(\pi_1(V), \ldots)$, a surgery problem with invariant κ of the form

$$\begin{array}{c}
(W^\kappa, \partial W^\kappa = V\times\{0\} \cup (\partial V \times I) \cup V') \\
\Big\downarrow \mathrm{sp} \\
(V\times I, \partial(V\times I) = V\times\{0\} \cup (\partial V \times I) \cup V\times\{1\})
\end{array}$$

such that sp maps the part $V\times\{0\} \cup (\partial V \times I)$ of ∂W^κ identically to the corresponding part of $\partial(V\times I)$, and maps V' to $V\times\{1\}$ by a simple homotopy equivalence. (The map sp has degree 1, of course, and a stable trivialization of $\tau_W \oplus \mathrm{sp}^*(\gamma_{V\times I})$ is also given. Further, an orientation is needed somewhere; to get it, assume, as always, that B is connected, consider $L_{2k}(\pi_1(B), \pi_1(f))$ as a globally constant sheaf on B^{or}, as in I.5, and fix a point in B^{or}.) This surgery problem is essentially determined by its surgery invariant κ. At any rate, the map

$$\begin{array}{rcl}
\lambda_V: L_{2k}(\cdots) & \longrightarrow & \mathcal{S}(V) \\
\kappa & \longmapsto & \text{equivalence class of } (W^\kappa; V \cong V\times\{0\}, V')
\end{array}$$

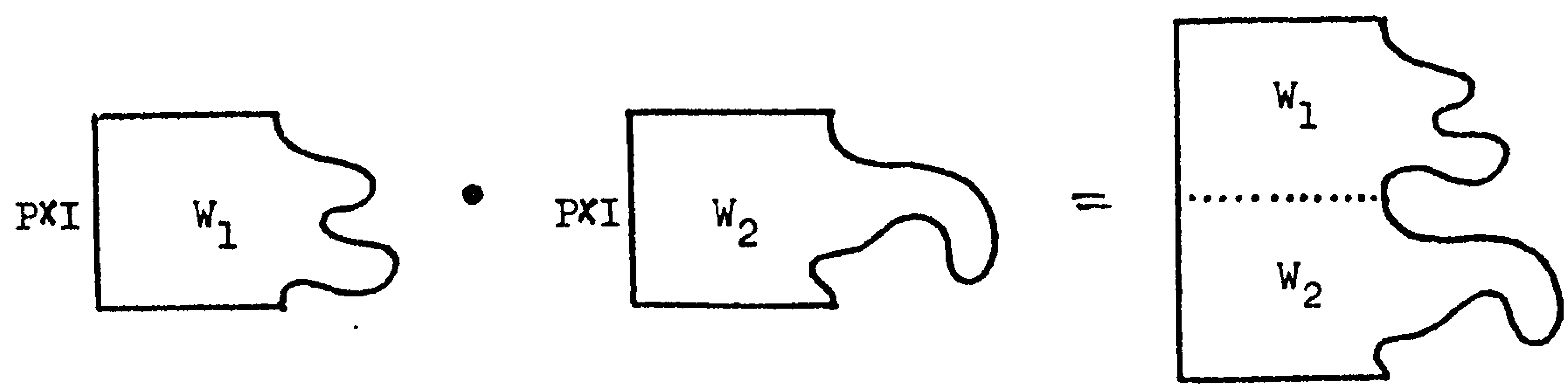
is well defined.

I.7 PROPOSITION: $(z, \kappa) \in \pi_{2k}(M(B,f)) \oplus L_{2k}(\pi_1(B), \pi_1(f))$ belongs to K (from I.4) precisely if $\delta_V(z) = \lambda_V(\kappa)$.

I.8 REMARKS:

(i) In a sense, promise I.4 could be fulfilled (because of I.7) with $S_{2k}(B,f) := \mathcal{S}(V)$, $\delta = \delta_V$, $\lambda = \lambda_V$. But then $S_{2k}(B,f)$ would be neither well defined, nor functorial, nor a group.

(ii) Still, $\mathcal{S}(V)$ can be given a monoid structure in various ways. V admits a factorization $V \cong P \times I$, where P^{2k-2} is a (B,f)-manifold with boundary. (This is true for dimension reasons, 'just about'; note that the factorization is not unique.) Such a factorization determines a multiplication on $\mathcal{S}(V)$, and δ_V, λ_V become homomorphisms: Given two elements of $\mathcal{S}(V)$, glue the first 'on top' of the second to get their product in $\mathcal{S}(V)$. Picture:



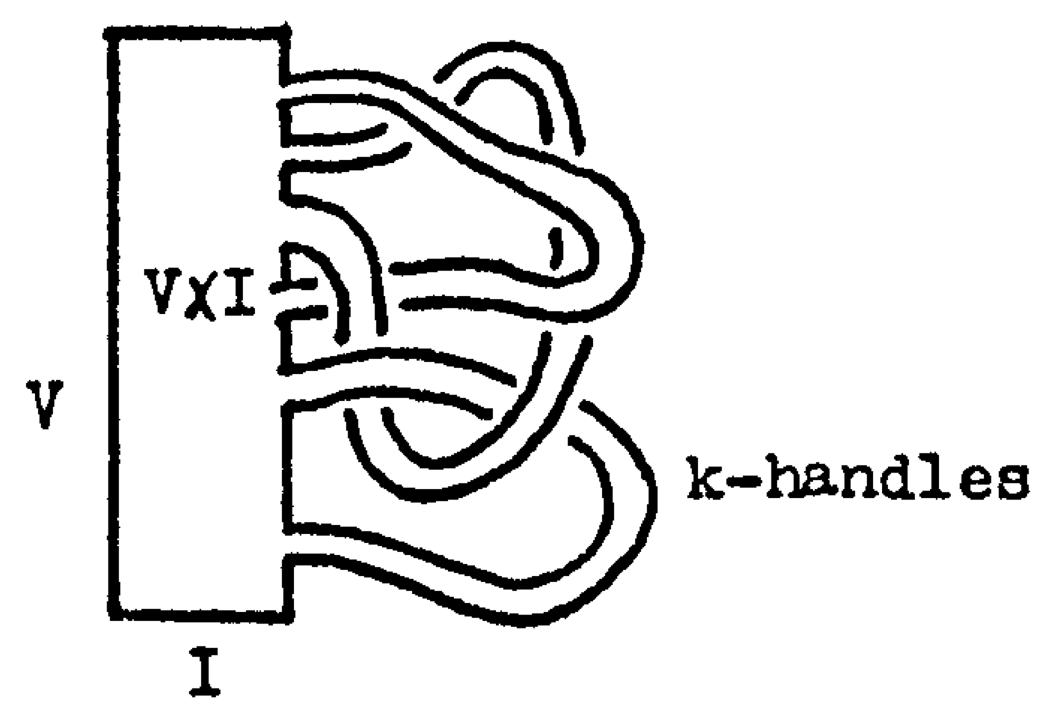
(iii) Every element of $\mathcal{S}(V)$ has a representative $(W;V,V')$ such that the classifying map $W \longrightarrow B$ (for the normal bundle) is k-connected (start with any representative, and perform surgeries on spheres of dimension $<k$ to kill $\pi_i(B,W)$ for $i \leq k$; see [WALL 1, Thm.1.2]).
For some elements of $\mathcal{S}(V)$ – not all – we can do better. Suppose e.g. that $x \in \mathcal{S}(V)$ is in the image of δ_V or λ_V. Let $(W;V,V')$ be a representative for x, and suppose that the map $W \longrightarrow B$ is already k-connected; then, by construction of V, the inclusion $V \hookrightarrow W$ is $(k-1)$-connected. Moreover, the inclusion $V' \hookrightarrow W$ is also $(k-1)$-connected (since $V' \cong V$ in the present case). So we have $H_i(W,V) = 0$ for $i<k$,

but also $H_i(W,V) = H^{2k-i}(W,V') = 0$ for $i > k$, for any coefficient sheaf on W.

Now a standard argument [WALL 1, Lemma 2.3] shows that $H_k(W,V)$ is stably free and based (coefficients $\mathbb{Z}[\pi_1(W)]$); if it is not yet free, a slight modification of W (namely surgery on trivial $(k-1)$-spheres) will make it so. If it is free, another standard argument shows that W can be written as

$$W = V \times I \cup (\text{handles of index } k),$$

as in the picture:



The attaching maps for the k-handles, $S^{k-1} \times D^k \hookrightarrow V$, are almost determined by their homotopy class (in $\pi_k(B,V) = H_k(B,V;\mathbb{Z}[\pi_1(B)])$; see [WALL 1, Thm.1.1]). 'Almost' means 'up to regular homotopy of immersions'; in the present case, the difference between 'regular homotopy of immersions' and 'isotopy of embeddings' should not strike one with horror.

To summarize, it transpires that certain elements in $\mathcal{S}(V)$ can be codified in an algebraic way.

PROOF of I.7.

First half: Suppose $z \in \pi_{2k}(M(B,f))$, $\kappa \in L_{2k}(\pi_1(B),\pi_1(f))$, and $\sigma_V(z) = \lambda_V(\kappa)$ in $\mathcal{S}(V)$. We must show that $(z,\kappa) \in K$. – Let $(W^\kappa;V,V')$ be a representative for $\lambda_V(\kappa)$, obtained as in I.6(iv).

So $(W^K;V,V')$ is just the upper half of a surgery problem:

$$\begin{array}{c}(W^K;V,V')\\ \big\downarrow{\scriptstyle sp}\\ (V\times I;\ V\times\{0\},\ V\times\{1\})\end{array} \quad ,$$

as in I.6(iv) .

Claim: The equality $\sigma_V(z)=\lambda_V(K)$ yields an s-cobordism of (B,f)-manifolds $(W_3;V,V')$ from V to V' (modulo boundary). For $\sigma_V(z)$ is represented by a cobordism of the form $(V\times I\cup N^{2k};\ V\times\{0\},\ V\times\{1\})$, and the claim follows from the definition of 'equivalence' (I.6(ii)) .

Applying the s-cobordism theorem gives a diffeomorphism $\hat{\alpha}: V \longrightarrow V'$, modulo boundary again, which 'preserves' most of the structure in sight (this will be explained and appealed to later on).

Using the map sp as a transporting agent, we may transport the diffeomorphism $\hat{\alpha}$ downwards to obtain a simple homotopy equivalence $\alpha : V\cong V\times\{0\} \longrightarrow V\times\{1\}\cong V$, restricting to the identity on the boundary. We now let

$$X := \text{'open book defined by } \alpha\text{'}$$

$$= V\times I/_{(v,0)\sim(\alpha(v),1)} \cup D^2\times\partial V \quad ,$$

where the two parts of this union are to be glued together along their common boundary $S^1\times\partial V$.

(Assume that V is triangulated, and that α is a simplicial map, still restricting to the identity on the boundary. Then X is a finite Poincaré complex; moreover, with a bit of labour, it can be arranged to have no k-cells, since it is the union of two copies of $V\times I$ glued together by a simple homotopy equivalence of the boundaries, and since V has a handle decomposition with handles of index $< k$ only.)

Similarly we put

$$N := W^K/_{v\in V \sim \hat{\alpha}(v)\in V'} \cup D^2\times\partial V \quad ,$$

again glueing together along the common boundary $S^1\times\partial V$.

Then we have an obvious surgery problem $N \longrightarrow X$ (here the structure-preserving properties of $\hat{\alpha}$ and α are needed, for it is otherwise not clear where the bundle on X should come from). It has surgery obstruction κ , and $[N] = z$ by construction; also, X has no k-cells.

Unfortunately, a longish argument is needed to show that the semi-torsion of X is 0 . X was obtained by glueing together two copies of $V\times I$, say $(V\times I)_0$ and $(V\times I)_1$, along their boundaries , using a simple homotopy equivalence

$$\beta : \partial(V\times I)_0 \longrightarrow \partial(V\times I)_1$$

(so that X is a 'twisted double').

Fix a handle decomposition for $V\times I$ with handles of index $< k$ only . There are now two distinct ways to get 'relevant' cell decompositions for X without k-cells (from the given handle decomposition).

To obtain these, note that the handle decomposition gives a simple homotopy equivalence

$$V\times I \longrightarrow X_- \quad ,$$

where X_- is a (finite) CW-complex of dimension $< k$; but, using dual handles, it also yields a simple homotopy equivalence of relative CW-complexes

$$(V\times I, \partial(V\times I)) \longrightarrow (X_+, \partial(V\times I)) \quad ,$$

where X_+ is obtained by (successively) attaching cells of dimension $> k$

to $\partial(V\times I)$ etc. .

Now X is the pushout of the diagram

$$\begin{tikzcd}
(V\times I)_1 \\
\partial(V\times I)_1 \arrow[u, hook] \\
\partial(V\times I)_0 \arrow[u, "\beta"] \arrow[r, hook] & (V\times I)_0
\end{tikzcd},$$

but X is also simple homotopy equivalent to the pushout of the diagram

$$\begin{tikzcd}
(V\times I)_1 \\
\partial(V\times I)_1 \arrow[u, hook] \arrow[r, "\beta^{-1}"] & \partial(V\times I)_0 \arrow[r, hook] & (V\times I)_0
\end{tikzcd}.$$

Drawing inspiration from the former diagram, we obtain a simple homotopy equivalence e_1 of X with the pushout X_1 of the diagram

$$\begin{tikzcd}
X_- \\
(V\times I)_1 \arrow[u, "\cong"] \\
\partial(V\times I)_1 \arrow[u, hook] \\
\partial(V\times I)_0 \arrow[u, "\beta"] \arrow[r, hook] & X_+
\end{tikzcd};$$

and drawing inspiration from the latter, a simple homotopy equivalence e_2 of X with the pushout X_2 of the diagram

$$\begin{tikzcd} X_+ \\ \partial(V\times I)_1 \arrow[u, hook] \arrow[r, "\beta^{-1}"] & \partial(V\times I)_0 \arrow[r, hook] & (V\times I)_0 \arrow[r, "\simeq"] & X_- \end{tikzcd} .$$

Both pushouts X_1 and X_2 are CW-complexes with no k-cells. Let $C(\tilde{X}_1)$, $C(\tilde{X}_2)$ be the associated chain complexes (which are f.g. free and based over $\mathbb{Z}[\pi_1(X)]$). We shall prove that both X_1 and X_2 have semi-torsion 0 by proving

(i) that the map (of chain complexes) $(e_2 \cdot e_1^{-1})_* : C(\tilde{X}_1) \longrightarrow C(\tilde{X}_2)$ has semi-torsion 0 ;

(ii) that the composite map

$$C(\tilde{X}_1)^* \longrightarrow C(\tilde{X}_1) \xrightarrow[(e_2 \cdot e_1^{-1})_*]{} C(\tilde{X}_2)$$

has semi-torsion 0 too (the left arrow is 'cap product with the fundamental class').

(Both maps split into halves, so the semi-torsion terminology makes sense.) Actually, the two assertions (i) and (ii) are supposed to be obvious. By construction, the cell decompositions for the simple homotopy type of X ,

$$e_1 : X \longrightarrow X_1 \qquad \text{and} \qquad e_2 : X \longrightarrow X_2 \quad ,$$

are 'dual' to each other (cf. [WALL 1,Thm.2.1]) ,which proves (ii) .

(In greater detail, the diagrams

$$\begin{tikzcd} X_- \arrow[r, hook, "i"] \arrow[d, "\cong"'] & X_1 \arrow[d, "\cong"', "e_1^{-1}"] \\ (V\times I)_1 \arrow[r, hook] & X \end{tikzcd}$$

and

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "e_2"', "\wr"] & X/(V\times I)_0 = (V\times I)_1/\partial(V\times I)_1 \arrow[d, "\wr"] \\
X_2 \arrow[r, "p"'] & X_+/\partial(V\times I)_1
\end{tikzcd}$$

are homotopy commutative. Then we may conclude that the diagram

$$(\ast)\qquad \begin{tikzcd}
C(X_-)^* \arrow[r, "y"'] & C(X_+)/C(\partial(V\times I)_1) \\
C(X_1)^* \arrow[u, "i^*"] \arrow[r, "z"'] & C(X_2) \arrow[u, "p_*"']
\end{tikzcd}$$

is chain homotopy commutative, with z being the chain map in assertion (ii), and y the composite

$$C(X_-)^* \cong C((V\times I)_1)^* \xrightarrow{\text{Poincare duality}} C((V\times I)_1)/C(\partial(V\times I)_1) \cong C(X_+)/C(\partial(V\times I)_1)\ ,$$

for some cell structure on $(V\times I)_1$. All chain maps in the definition of y are simple homotopy equivalences, hence so is y itself; assertion (ii) follows therefore from diagram $(\ast)$.)

For assertion (i) concerning the semi-torsion of the chain map $(e_2\cdot e_1^{-1})_*$, note that the diagram

$$\begin{tikzcd}
 & X_1 \arrow[dr, "e_1^{-1}"] & \\
X_- \arrow[ur] \arrow[dr] & & X \\
 & X_2 \arrow[ur, "e_2^{-1}"'] &
\end{tikzcd}$$

is commutative up to homotopy. Hence the map $e_2\cdot e_1^{-1}: X_1 \longrightarrow X_2$

can be deformed into one which induces a cellular homoemorphism of the (k-1)-skeletons (both of which are copies of X_-). This proves (i) , and completes the first half of the proof of I.7 .

Second half.

Suppose we have the appropriate surgery problem:

$$\begin{tikzcd} N^{2k} \arrow[d, "sp"'] \arrow[dr] & \\ X \arrow[r] & B \end{tikzcd}$$

(where X has no k-cells and semi-torsion 0). We may suppose that (like B) X is connected, and that $\pi_1(X) \longrightarrow \pi_1(B)$ is iso (as in [WALL 1, Thm.9.4 or Lemma 2.8]) ; also, that the (k-1)-skeleton of X is a simplicial complex. Performing surgery if necessary, we may further assume that $\pi_i(sp) = 0$ for $i \leq k$; then $\pi_{k+1}(sp) = H_{k+1}(sp)$ $= \ker [H_k(N) \longrightarrow H_k(X)]$, which is equal to $H_k(N)$ in the present case (coefficients $\mathbb{Z}[\pi_1(X)]$; cf. [WALL 1,ch.5]) . This is stably free and based, and equipped with intersection numbers as well as self-intersection numbers; in other words, it represents an element in $L_{2k}(\pi_1(X),w) = L_{2k}(\pi_1(B),\pi_1(f))$, namely κ .

Now let X_- be the (k-1)-skeleton of X . The inclusion $X_- \hookrightarrow X$ can be factored through N (up to homotopy):

$$X_- \xrightarrow[j]{} N \xrightarrow[sp]{} X \quad .$$

Moreover, j can be supposed to be a 'nice' embedding (since X_- is a polyhedron); let N_- be a regular neighbourhood of $j(X_-)$, and $N_+ = N - \mathrm{int}(N_-)$.

For dimension reasons N_- can be written as a product $V_1 \times I$ (where V_1 is a (B,f)-manifold with boundary, of dimension 2k-1).

Thus we may consider $(N_+ ; V_1 \times \{0\}, V_1 \times \{1\})$ as a cobordism modulo boundary from $V_1 \cong V_1 \times \{0\}$ to something else (compare [QUINN]). Clearly, this cobordism represents $\delta_{V_1}(z) \in \mathcal{S}(V_1)$ (if we put $z = [N] \in \pi_{2k}(M(B,f))$).

(Here, of course, $\mathcal{S}(V_1)$, δ_{V_1} and λ_{V_1} are defined as in I.6(ii),(iii),(iv) , with V replaced by V_1 .)

On the other hand, the same cobordism can be directly seen to represent $\lambda_{V_1}(\kappa) \in \mathcal{S}(V_1)$ as well. To see this, we analyze $(N_+ ; V_1 \times \{0\}, V_1 \times \{1\})$ as in I.8(iii) . By the usual argument , $H_k(N_+, V_1 \times \{0\})$ is (stably) free and based, and any basis gives us a decomposition of N as

$$V_1 \times I \cup k\text{-handles}$$

(provided it is a preferred basis). So, to prove the claim, it suffices to show that the two homomorphisms in the diagram

$$\begin{tikzcd}[row sep=small] & H_k(N_+, V_1) \\ H_k(N_+) \arrow[ur] \arrow[dr] & \\ & H_k(N) \cong H_{k+1}(sp) \end{tikzcd}$$

are both isomorphisms, and that the resulting isomorphism

$$H_k(N) \longrightarrow H_k(N_+, V_1)$$

preserves the preferred bases.

The upper homomorphism is isomorphic because, in the decomposition of N_+ as $V_1 \times I \cup k$-handles , all the attaching maps for the k-handles are nulhomotopic (and because V_1 is 'essentially' $(k-1)$-dimensional) . (Proof: By restricting sp to N , obtain a map of pairs

$$(N_+, V_1 \times I) \longrightarrow (k\text{-skeleton of } X,\ (k-1)\text{-skeleton of } X)$$

which induces the given identification $V_1 \times I = N_- \cong X_-$ $= (k-1)$-skeleton of X ; then remember that X has no k-cells .)

The lower homomorphism, $H_k(N_+) \longrightarrow H_k(N)$, is clearly surjective.

The diagram

$$\begin{array}{ccc} H^{k-1}(N) & \longrightarrow & H^{k-1}(N_-) \\ \cong\uparrow & & \uparrow\cong \\ H^{k-1}(X) & \xrightarrow{\cong} & H^{k-1}(X_-) \end{array}$$

shows that $H^{k-1}(N) \longrightarrow H^{k-1}(N_-)$ is an isomorphism; hence so is $H_{k+1}(N) \longrightarrow H_{k+1}(N,N_+)$ (by Poincare duality). The exact homology sequence of the pair (N,N_+) then shows that $H_k(N_+) \longrightarrow H_k(N)$ is injective, as required.

It remains to prove the assertion concerning compatibility of the two bases, or let us say: equality of the two (classes of) bases b_1 and b_2 , b_1 being the basis which originally belonged to $H_k(N) \cong H_{k+1}(sp)$, and b_2 that coming from $H_k(N_+;V_1\times\{0\})$. Again, this requires a painfully long argument, which the reader may want to skip.

The free module $H_k(N)$, together with mutual and self-intersection numbers, and with the basis b_1 , determines an element in $L_{2k}(\pi_1(B),\pi_1(f))$; if the basis b_2 is used instead, another element. The first is κ ; call the second κ' .

It is clear that there exists a surgery problem of the form

$$\begin{array}{c} (N_+;V_1\times\{0\},V_1\times\{1\}) \\ \big\downarrow \\ (V_1'\times I;V_1'\times\{0\},V_1'\times\{1\}) \end{array}$$

with surgery obstruction κ' . (Here V_1' is just a copy of V_1 ; the map restricts to a simple homotopy equivalence between the boundaries ∂N_+ and $\partial(V_1'\times I)$; in fact, it gives the obvious identification off, but not on, $V_1 \times \{1\} \subset \partial N_+$.)

Let X_1 be obtained by removing the interior of N_+ from N and glueing in $V_1' \times I$ instead, using the simple homotopy equivalence $\partial N_+ \longrightarrow V_1' \times I$ just described.

Then we get a surgery problem

$$\begin{array}{c} N \\ \Big\downarrow{\scriptstyle sp_1} \\ X_1 \end{array}$$

by trivially extending the 'relative' surgery problem $N_+ \longrightarrow V_1' \times I$ mentioned above; it still has surgery obstruction K' .

X_1 is a finite Poincare complex, an open book (or 'twisted double') again; for $N - \mathrm{int}(N_+) = N_- = V_1 \times I$.

It follows (see the end of the first half of this proof) that its simple homotopy type has a 'preferred' cell decomposition with no k-cells and semi-torsion 0 ; in fact any handle decomposition of $N_- = V_1 \times I$ determines such a cell decomposition (or maybe two such) , and we do have a preferred handle decomposition for N_- since N_- is the regular neighbourhood of a polyhedron .

By considering the map $sp_1 : N \longrightarrow X_1$ as a collapsing map (which collapses certain k-handles in N) , one finds that the diagram

$$\begin{tikzcd} & N \arrow[dl, "sp_1"'] \arrow[dr, "sp"] & \\ X_1 & & X \end{tikzcd} \quad \text{can be completed to} \quad \begin{tikzcd} & N \arrow[dl, "sp_1"'] \arrow[dr, "sp"] & \\ X_1 \arrow[rr, "f"'] & & X \end{tikzcd} .$$

Claim: $f : X_1 \longrightarrow X$ is a simple homotopy equivalence.

This is so because both X_1 and X have no k-cells and semi-torsion 0 , and because the bottom half of f is, or can be deformed into, a cellular homeomorphism .

It should now be obvious that the two bases b_1 , b_2 are equal.

What we have shown now (starting from the existence of a certain surgery problem) is that $\sigma_{V_1}(z) = \lambda_{V_1}(\kappa)$ in $\mathcal{S}(V_1)$; what we need is

$$\sigma_V(z) = \lambda_V(\kappa) \quad \text{in} \quad \mathcal{S}(V) \ .$$

Choose an embedding $V_1 \hookrightarrow V$ such that the γ-structure (or (B,f)-structure) on V_1 obtained by restricting that on V is concordant to the original γ-structure on V_1 . (This is possible by Hirsch's immersion theorem; see [WALL 1, ch.1, Proposition] .)

The embedding will induce a map $\mathcal{S}(V_1) \longrightarrow \mathcal{S}(V)$ making the diagram

$$\begin{tikzcd}
 & \mathcal{S}(V_1) \arrow[dd] & \\
\pi_{2k}(M(B,f)) \arrow[ur, "\sigma_{V_1}"] \arrow[dr, "\sigma_V"'] & & L_{2k}(\pi_1(B), \pi_1(f)) \arrow[ul, "\lambda_{V_1}"'] \arrow[dl, "\lambda_V"] \\
 & \mathcal{S}(V) &
\end{tikzcd}$$

commutative . This completes the proof.

II. ALGEBRAIC PRELIMINARIES

The description of the obstruction group $S_{2k}(B,f)$ associated with a fibration $f:B \longrightarrow BO$ requires two steps. The first of these is a 'distillation procedure' which will leave us with an algebraic object containing all the necessary information (essentially this is the – singular or cellular – chain complex of the universal covering space of B, together with some additional structure). The second step consists in the construction of the groups $S_{2k}(B,f)$ (simultaneously for all integers k) as invariants of this algebraic object.
Only the last step will be dealt with in this section.

Let π be a group with 'orientation' homomorphism $w:\pi \longrightarrow Z_2$, $\Lambda := \mathbb{Z}[\pi]$ the group ring. As usual, this is considered as a ring with involution (or involutory anti-automorphism), the involution '$-$' being given by

$$(\Sigma\, n(g)\cdot g)^- = \Sigma (-1)^{w(g)}\cdot n(g)\cdot g^{-1}$$

(cf. [WALL 1,p.21]).

If M is a left Λ-module, a 'sesquilinear form' on M is a map

$$s:\ M\times M \longrightarrow \Lambda$$

which is biadditive and satisfies $s(ax,by) = a\cdot s(x,y)\cdot \bar{b}$ for x,y in M and a,b in Λ. (This and the following section owe very much to [QUINN] , where it is explained how these not-necessarily-symmetric sesquilinear forms arise in topology. See also [WALL 1,p.260 and p.2] .)

The sesquilinear forms on M form an abelian group $\mathrm{Sel}(M)$; the map

$$T:\ \mathrm{Sel}(M) \longrightarrow \mathrm{Sel}(M)$$

given by $\quad T(s)(x,y) := (s(y,x))^{-}$ $\quad$ is an involutory automorphism.

II.1 CONSTRUCTION:

Let $\quad C = \cdots - C_{-1} \longleftarrow C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \leftarrow \cdots$

be a chain complex of Λ-modules. We shall use it to construct a new chain complex (of abelian groups, and with period two):

$$Q(C): \qquad \begin{tikzcd} Q(C)_0 \arrow[r, bend right, "\delta_0"'] & Q(C)_1 \arrow[l, bend right, "\delta_1"'] \end{tikzcd}$$

Here $\quad Q(C)_0 = Q(C)_1 = \prod_{n\in\mathbb{Z}} \mathrm{Sel}(C_n)$, and δ_0 , δ_1 are defined by

$$\left[\delta_0((\lambda_n)_{n\in\mathbb{Z}})\right]_j \quad := \quad \lambda_{j-1}\cdot\partial_j^2 \; - \; \left[\lambda_j - (-1)^j T(\lambda_j)\right]$$

and $$\left[\delta_1((\lambda_n)_{n\in\mathbb{Z}})\right]_j \quad := \quad \lambda_{j-1}\cdot\partial_j^2 \; + \; \left[\lambda_j + (-1)^j T(\lambda_j)\right] \quad .$$

Thus a cycle in $Q(C)_0$ is a sequence of sesquilinear forms

$\lambda_i : C_i \times C_i \longrightarrow \Lambda$ $\qquad$ such that, for each i , the 'symmetrization of λ_i ' $\quad (= [\lambda_i - (-1)^i T(\lambda_i)])$ equals $\quad \lambda_{i-1}\cdot\partial_i^2 \qquad$ (i.e. the composite $\quad C_i \times C_i \longrightarrow C_{i-1}\times C_{i-1} \xrightarrow[\lambda_{i-1}]{} \Lambda$).

Note that Q is a contravariant functor .

The point of view underlying the categorical manipulations below is that a cycle in $Q(C)_0$ is to the chain complex C what a bundle over a space is to that space. (This will be made clear in the next section, I hope.) One weakness of the analogy is that there is no homotopy lifting property on the algebraic side; some efforts are therefore still necessary, and will be made presently. (Further problems, which I have not fathomed yet, arise in connection with the Whitney sum. It is very possible that these can be solved .)

II.2 ABSTRACT NONSENSE.

Let $\mathcal{C}_\Lambda$ be the category of chain complexes

$$C : \quad \cdots \longleftarrow C_{-1} \longleftarrow C_0 \longleftarrow C_1 \longleftarrow \cdots$$

having the following properties:

(i) each C_i is a finitely generated stably free Λ-module with a preferred equivalence class of s-bases (as in [WALL 1]);

(ii) there exists an integer i_0 such that $C_i = 0$ for $i < i_0$.

The morphisms in $\mathcal{C}_\Lambda$ are the chain maps.

Second, let $\mathcal{H}\mathcal{C}_\Lambda$ have the same objects as $\mathcal{C}_\Lambda$, but homotopy classes of chain maps as morphisms.

Third, let $\tilde{\mathcal{C}}_\Lambda$ be the category whose objects are pairs (C,λ), where C is a chain complex belonging to $\mathcal{C}_\Lambda$, and $\lambda = (\lambda_n)_{n\in\mathbb{Z}}$ is a cycle in $Q(C)_0$. A morphism from (C,λ) to (C',λ') shall be a chain map $f: C \longrightarrow C'$ such that $f^*: Q(C') \longrightarrow Q(C)$ maps λ' to λ.

Forgetting the cycles gives a forgetful functor $\tilde{\mathcal{C}}_\Lambda \longrightarrow \mathcal{C}_\Lambda$.

Now suppose that $f: (C,\lambda) \longrightarrow (C',\lambda')$ is a morphism in $\tilde{\mathcal{C}}_\Lambda$ such that the underlying chain map $C \longrightarrow C'$ is a homotopy equivalence, or in other words, an isomorphism in $\mathcal{HC}_\Lambda$. We can then hardly say anything useful about f , just because there is no 'homotopy lifting theorem' . However, we can complete the diagram

$$\begin{array}{ccc} \tilde{\mathcal{C}}_\Lambda & & \\ \downarrow & & \\ \mathcal{C}_\Lambda & \longrightarrow & \mathcal{HC}_\Lambda \end{array}$$

to a square

$$\begin{array}{ccc} \tilde{\mathcal{C}}_\Lambda & \longrightarrow & \mathcal{H}\tilde{\mathcal{C}}_\Lambda \\ \downarrow & & \downarrow \\ \mathcal{C}_\Lambda & \longrightarrow & \mathcal{HC}_\Lambda \end{array}$$

by defining $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$ to be the quotient category obtained from $\tilde{\mathcal{C}}_\Lambda$ by making all morphisms in $\tilde{\mathcal{C}}_\Lambda$ invertible whose underlying chain maps are homotopy equivalences. ($\mathcal{H}\tilde{\mathcal{C}}_\Lambda$ is characterized by a universal property; see [GABRIEL-ZISMAN] for information on quotient categories.) Replacing $\tilde{\mathcal{C}}_\Lambda$ by $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$ solves the problem above, or rather circumvents it.

The remainder of this section gives the description of a functor from $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$ to the category $\mathcal{AG}$ of abelian groups.

II.3 A FUNCTOR $\mathcal{H}\tilde{\mathcal{C}}_\Lambda \longrightarrow \mathcal{AG}$.

(i) Technicalities.

Let $f,g\colon C \longrightarrow D$ be two chain maps in $\mathcal{C}_\Lambda$. For a fixed integer k, let us say that f is 'k-homotopic' to g if there exists a chain homotopy $(s_n : C_{n-1} \longrightarrow D_n)_{n\in\mathbb{Z}}$ from f to g such that $s_n = 0$ for $n \leqslant k$. Then 'k-homotopy' is an equivalence relation compatible with composition of morphisms; so we obtain a new category $\mathcal{H}_k\mathcal{C}_\Lambda$ with the same objects as $\mathcal{C}_\Lambda$, but k-homotopy classes of chain maps as morphisms. The various $\mathcal{H}_k\mathcal{C}_\Lambda$, fit into a row of categories and functors

$$\mathcal{H}\mathcal{C}_\Lambda \leftarrow \cdots - \quad \mathcal{H}_{k-1}\mathcal{C}_\Lambda \longleftarrow \mathcal{H}_k\mathcal{C}_\Lambda \longleftarrow \mathcal{H}_{k+1}\mathcal{C}_\Lambda \leftarrow \cdots - \quad \mathcal{C}_\Lambda \quad ;$$

two morphisms in $\mathcal{C}_\Lambda$ are equal precisely if they are k-homotopic for all k ,and are homotopic precisely if they are k-homotopic for some k (since complexes in $\mathcal{C}_\Lambda$ are bounded below).

We can similarly define categories $\mathcal{H}_k\widetilde{\mathcal{C}}_\Lambda$, in fact a whole diagram

$$\begin{array}{ccccccccc}
\mathcal{H}\widetilde{\mathcal{C}}_\Lambda \leftarrow \cdots - & \mathcal{H}_{k-1}\widetilde{\mathcal{C}}_\Lambda & \longleftarrow & \mathcal{H}_k\widetilde{\mathcal{C}}_\Lambda & \longleftarrow & \mathcal{H}_{k+1}\widetilde{\mathcal{C}}_\Lambda & \leftarrow \cdots - & \widetilde{\mathcal{C}}_\Lambda \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{H}\mathcal{C}_\Lambda \leftarrow \cdots - & \mathcal{H}_{k-1}\mathcal{C}_\Lambda & \longleftarrow & \mathcal{H}_k\mathcal{C}_\Lambda & \longleftarrow & \mathcal{H}_{k+1}\mathcal{C}_\Lambda & \leftarrow \cdots - & \mathcal{C}_\Lambda
\end{array}$$

using the same procedure as before; that is, we obtain $\mathcal{H}_k\widetilde{\mathcal{C}}_\Lambda$ by making all morphisms in $\widetilde{\mathcal{C}}_\Lambda$ invertible whose underlying chain maps are k-homotopy equivalences.

An object (C,λ) in $\widetilde{\mathcal{C}}_\Lambda$ will be called 'k-regular' if the k-th component of λ , namely $\lambda_k : C_k \times C_k \longrightarrow \Lambda$, is a simple nondegenerate sesquilinear form; in other words, if its adjoint homomorphism (from the left Λ-module C_k to the left Λ-module of anti-linear maps $C_k \longrightarrow \Lambda$) is a simple isomorphism .

If $f\colon (C,\lambda) \longrightarrow (C',\lambda')$ is a morphism in $\tilde{\mathcal{C}}_\Lambda$, and both domain and range are k-regular, then f embeds C_k in C'_k as an 'orthogonal direct summand' ; here 'orthogonal' can be taken to mean 'left orthogonal' or 'right orthogonal' (usually not both) , as required.

(ii) Resolutions.

Suppose that (C,λ) is an object in $\tilde{\mathcal{C}}_\Lambda$ such that $C_i = 0$ for $i<k$ and $H_*(C) = 0$ (so C is acyclic). Then C_k can be considered as carrying a $(-1)^k$-hermitian form . Namely, we have a sesquilinear form

$$\lambda_k\colon C_k \times C_k \longrightarrow \Lambda$$

which is $(-1)^k$-symmetric, i.e. $\lambda_k = (-1)^k \cdot T(\lambda_k)$ (because λ is a cycle in $Q(C)_0$); and we also have a function

$$\mu_k\colon C_k \longrightarrow \Lambda/\{\gamma-(-1)^k\cdot\bar{\gamma} \mid \gamma\in\Lambda\}$$

given by $\mu_k(x)$ = (coset of) $\lambda_{k+1}(\hat{x},\hat{x})$

(where $\hat{x}$ is any element mapping to x under $\partial\colon C_{k+1} \longrightarrow C_k$).

Again, μ_k is well defined because of the cycle condition on λ .

The pair λ_k, μ_k satisfies all the requirements in [WALL 1, Thm.5.2] , which define a $(-1)^k$-hermitian (or quadratic) form . (C,λ) may therefore be considered as some sort of (free, acyclic) <u>resolution</u> of this hermitian/quadratic form.

If we assume additionally that (C,λ) is k-regular, then λ_k and μ_k define a special (nonsingular) hermitian form on C_k. The Witt group of such forms is the surgery obstruction group $L_{2k}(\pi,w)$.

To summarize:

Definition and observation.

An object (C, λ) in $\tilde{\mathcal{E}}_\Lambda$ will be called a k-resolution if it is k-regular, if C is acyclic and if $C_i = 0$ for $i<k$.

Any k-resolution determines an element in $L_{2k}(\pi, w)$.

(iii) An auxiliary functor from $\tilde{\mathcal{E}}_\Lambda$ to *Groups*.

This functor $F_k : \tilde{\mathcal{E}}_\Lambda \longrightarrow$ *Groups* is defined, or characterized, by the following properties:

(a) F_k factors through $\mathcal{H}_k\tilde{\mathcal{E}}$; or equivalently, if $f: (C, \lambda) \longrightarrow (C', \lambda')$ is a morphism in $\tilde{\mathcal{E}}_\Lambda$ such that the underlying chain map $C \rightarrow C'$ is a k-homotopy equivalence, then $F_k(f)$ is an isomorphism of groups.

(b) $F_k(0) = L_{2k}(\pi, w)$ (0 being the trivial complex).

(c) There is given a rule R which to each k-regular object (C, λ) in $\tilde{\mathcal{E}}_\Lambda$ associates a 'characteristic element' $R((C, \lambda))$ in $F_k((C, \lambda))$. R satisfies the following conditions:

(α) If (C, λ) is a k-resolution determining an element κ in $L_{2k}(\pi, w)$, then $R((C, \lambda)) \overset{*)}{=} \kappa$;

---

*) Of course, $R((C, \lambda)) \in F_k((C, \lambda))$ and $\kappa \in L_{2k}(\pi, w)$; but these two groups may be identified since the trivial map $0 \longrightarrow (C, \lambda)$ induces an isomorphism $L_{2k}(\pi, w) = F_k(0) \longrightarrow F_k((C, \lambda))$ according to property (a) above.

(β) if (C,λ) , (C',λ') , (C'',λ'') are all k-regular and $f': (C',\lambda') \longrightarrow (C,\lambda)$, $f'': (C'',\lambda'') \longrightarrow (C,\lambda)$ are two morphisms in $\tilde{\mathcal{C}}_\Lambda$ such that C_k is the left orthogonal direct sum (see explanation below) of $f'(C_k')$ and $f''(C_k'')$, THEN

$$R((C,\lambda)) = F_k(f')\,R((C',\lambda')) \cdot F_k(f'')\,R((C'',\lambda'')) ,$$

where the dot $\cdot$ represents multiplication in $F_k((C,\lambda))$.

(d) The pair (F_k,R) is universal (= universally repelling, initial) among all pairs satisfying (a), (b) and (c) . That is , given another such pair (F_k',R') , a unique transformation $F_k \longrightarrow F_k'$ exists such that etc. etc. .

Alternatively, the properties (a'), (c'), (d') below also characterize F_k :

(a') Same as (a) above, i.e. F_k factors through $\mathcal{H}_k\tilde{\mathcal{C}}_\Lambda$.

(c') There is given a rule R , etc.etc. ; R satisfies the following conditions.

(α') If (C,λ) is a k-resolution determining the neutral element in $L_{2k}(\pi,w)$, then $R((C,\lambda)) = 1$.

(β') Same as (β) above.

(d') The pair (F_k,R) is universal among all pairs satisfying (a') , (c').

The second chracterization may be somewhat less pleasing; but it does have the advantage that a functor satisfying (a') , (c') and (d') can easily be constructed by means of generators and relations. A more explicit construction will be given in section IV (IV.10 , definition of 'F_k^{new} '); this will then be shown to satisfy the two sets of axioms above, (a),(b),(c),(d) as well as (a'),(c'),(d') (Proposition IV.12).

EXPLANATION: (of 'left orthogonal'):
Let X be a stably free f.g. Λ-module with a preferred class of s-bases, and suppose X carries a simple nondegenerate sesquilinear form. Let Y and Z be two submodules of X both stably free, f.g. and s-based (in their own right).
Say that X is the 'left orthogonal direct sum' of Y and Z if

- X is the direct sum of Y and Z ;
- Y is left orthogonal to Z , i.e. $y \cdot z = 0$ whenever $y \in Y$, $z \in Z$ (the dot denotes the sesquilinear form on X);
- the s-basis on X is the 'direct sum' of the two s-bases on Y and Z .

(iv) Change of rings.
Any commutative diagram

$$\begin{tikzcd} \pi \arrow[rr] \arrow[dr, "w"'] & & \pi' \arrow[dl, "w'"] \\ & Z_2 & \end{tikzcd}$$

of groups and homomorphisms induces a homomorphism of rings with involution from $\Lambda = \mathbb{Z}[\pi]$ to $\Lambda' = \mathbb{Z}[\pi']$, by means of which Λ' may be considered as a Λ-module .
Taking the tensor product with Λ' (over Λ) therefore yields a functor from $\mathcal{C}_{\Lambda}$ to $\mathcal{C}_{\Lambda'}$, another from $\tilde{\mathcal{C}}_{\Lambda}$ to $\tilde{\mathcal{C}}_{\Lambda'}$, yet another from $\partial\mathcal{H}\mathcal{C}_{\Lambda}$ to $\partial\mathcal{H}\tilde{\mathcal{C}}_{\Lambda'}$. and so on . (Problems with left and right module structures may occur, but they are not too serious.)

There is a corresponding relation between the functors $F_{k,\Lambda}$ and $F_{k,\Lambda'}$ from $\tilde{\mathcal{C}}_\Lambda$ and $\tilde{\mathcal{C}}_{\Lambda'}$ respectively to $\mathcal{G}roups$. (The additional subscripts are needed only to tell them apart; otherwise they are the functors defined in the previous sub-subsection.)

Indeed, the composite functor

$$\tilde{\mathcal{C}}_\Lambda \xrightarrow{\;-\otimes_\Lambda \Lambda'\;} \tilde{\mathcal{C}}_{\Lambda'} \xrightarrow{\;F_{k,\Lambda'}\;} \mathcal{G}roups$$

satisfies the two conditions (a') , (c') on p.II.8 . Hence, by the universal property (d') , there is a unique transformation from $F_{k,\Lambda}$ to $F_{k,\Lambda'} \cdot -\otimes_\Lambda \Lambda'$ having such and such properties.

(v) Kan adaptations.

Let $\mathcal{P}$ be a small category, $\mathcal{Q}$ any category, $E: \mathcal{P} \longrightarrow \mathcal{Q}$ a (covariant) functor, and $J: \mathcal{P} \longrightarrow \mathcal{S}ets$ a covariant functor to the category of sets.

By an ' E-adaptation of J ' (adaptation for short) will be meant a pair (A, ϕ) consisting of a covariant functor $A: \mathcal{Q} \longrightarrow \mathcal{S}ets$ and a natural transformation ϕ from $A \cdot E$ to J .

The adaptations of J form themselves a category; a morphism from (A_1, ϕ_1) to (A_2, ϕ_2) is a natural transformation $\psi: A_1 \longrightarrow A_2$ such that $\phi_1 = \phi_2 \cdot E^*(\psi)$ (where $E^*(\psi)$ is the transformation from $A_1 \cdot E$ to $A_2 \cdot E$ obtained by composing ψ with E).

OBSERVATION: The category of E-adaptations of J has a final object.

I propose to call this universal adaptation $\bar{J}$ the 'Kan adaptation' (provided it has no other name yet) , since it generalizes the usual

'Kan extension' . In other words, if $E\colon \mathcal{P} \longrightarrow \mathcal{Q}$ is the inclusion of a subcategory, then $\bar{J}$ is the Kan extension of J (see the appendix to [DOLD]).

Sketch proof of observation: Let X be an object of $\mathcal{Q}$, and write $[X{\downarrow}\mathcal{Q}\,;E]$ for the 'category of $\mathcal{Q}$-objects under X with an E-lifting' . (Its objects are pairs (Y,f) , where Y is an object of $\mathcal{P}$, and $f\colon X \longrightarrow E(Y)$ is a morphism in $\mathcal{Q}$; a morphism in $[X{\downarrow}\mathcal{Q}\,;E]$ from (Y,f) to (Y',f') is the same as a morphism $\alpha\colon Y \longrightarrow Y'$ in $\mathcal{P}$ making the $\mathcal{Q}$-diagram

$$\begin{tikzcd} E(Y) \arrow[rr, "E(\alpha)"'] & & E(Y') \\ & X \arrow[ul, "f"] \arrow[ur, "f'"'] & \end{tikzcd}$$

commutative).

The functor J on $\mathcal{P}$ induces another functor $J_X\colon [X{\downarrow}\mathcal{Q}\,;E] \longrightarrow \mathcal{S}ets$ (send (Y,f) to the set $J(Y)$). J_X has an inverse limit (which is a set; its elements are the natural transformations from the constant one-point functor to J_X). Denote the inverse limit of J_X by $\bar{J}(X)$; this defines $\bar{J}$ on objects, etc. .

(vi) At last.

The long-awaited functor $\bar{F}_{k,\Lambda}\colon \mathcal{H}\tilde{\mathcal{C}}_\Lambda \longrightarrow \mathcal{AG}$ is defined to be the Kan adaptation of $F_{k,\Lambda}\colon \tilde{\mathcal{C}}_\Lambda \longrightarrow \mathcal{G}roups$ (with respect to the canonical functor $E\colon \tilde{\mathcal{C}}_\Lambda \longrightarrow \mathcal{H}\tilde{\mathcal{C}}_\Lambda$).

Of course, $\bar{F}_{k,\Lambda}$ should go from $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$ to the category of groups; but all the groups in question happen to be abelian, as asserted in the following proposition (whose proof can be found somewhere in section IV).

II.4 PROPOSITION:

(a) Let $\phi : \bar{F}_{k,\Lambda}\cdot E \longrightarrow F_{k,\Lambda}$ be the universal transformation. Then, for any object (C,λ) in $\tilde{\mathcal{C}}_\Lambda$, the homomorphism

$$\phi_{(C,\lambda)} : \bar{F}_{k,}\cdot E((C,\lambda)) \longrightarrow F_{k,\Lambda}((C,\lambda))$$

is injective, and its image is contained in the centre of $F_{k,\Lambda}((C,\lambda))$.

(b) Let
$$\begin{tikzcd} \pi \arrow[r] \arrow[dr, "w"'] & \pi' \arrow[d, "w'"] \\ & Z_2 \end{tikzcd}$$

be a commutative diagram of homomorphisms, as in II.3(iv), giving rise to a homomorphism of rings with involution $\Lambda \longrightarrow \Lambda'$.
Write $\Psi : F_{k,\Lambda} \longrightarrow F_{k,\Lambda'}\cdot - \otimes_\Lambda \Lambda'$ for the 'natural' natural transformation constructed in II.3(iv) ('change of rings'). Then, for any object (C,λ) in $\tilde{\mathcal{C}}_\Lambda$, the homomorphism

$$\Psi_{(C,\lambda)} : F_{k,\Lambda}((C,\lambda)) \longrightarrow F_{k,\Lambda'}((C,\lambda)\otimes_\Lambda \Lambda')$$

maps $\quad \mathrm{Im}(\phi_{(C,\lambda)}) \subset F_{k,\Lambda}((C,\lambda))$

to $\quad \mathrm{Im}(\phi_{(C,\lambda)\otimes_\Lambda\Lambda'}) \subset F_{k,\Lambda'}((C,\lambda)\otimes_\Lambda\Lambda')$.

Hence $\bar{F}_{k,\Lambda}$ and $\bar{F}_{k,\Lambda'}$ are related by means of a canonical transformation from $\bar{F}_{k,\Lambda}$ to $\bar{F}_{k,\Lambda'}\cdot - \otimes_\Lambda \Lambda'$.

II.5. OBSERVATION: Write L_{2k} for the constant functor which to every object in $\tilde{\mathcal{C}}_\Lambda$ associates the group $L_{2k}(\Lambda)$ $(= L_{2k}(\pi,w)$ if desired), and to every morphism the identity. II.3(iii) gives a transformation

$L_{2k} \longrightarrow F_k$; for a given object (C,λ) it equals the homomorphism

$$F_k(0) = L_{2k}(\Lambda) \longrightarrow F_k((C,\lambda))$$

induced by the (only) morphism $0 \longrightarrow (C,\lambda)$.

The universal property of $\bar{F}_k$ gives a canonical factorization

$$\begin{tikzcd} & \bar{F}_k \arrow[d] \\ L_{2k} \arrow[ur] \arrow[r] & F_k \end{tikzcd}$$

Another cause for concern is the transfer.

Again let $\begin{tikzcd} \pi \arrow[r] \arrow[dr] & \pi' \arrow[d] \\ & Z_2 \end{tikzcd}$ be a diagram as in II.4(b) , but

assume additionally that $\pi \longrightarrow \pi'$ is the inclusion of a subgroup of finite index. Then every f.g. free and based module over $\Lambda' = \mathbb{Z}[\pi']$ can be regarded as one over $\Lambda = \mathbb{Z}[\pi]$; in this way a forgetful functor $?: \tilde{\mathcal{C}}_{\Lambda'} \longrightarrow \tilde{\mathcal{C}}_{\Lambda}$ is obtained.

II.6 PROPOSITION: Let

$$\text{transfer: } F_{k,\Lambda'} \longrightarrow F_{k,\Lambda} \cdot ?$$

be the natural transformation determined by the universal property II.3(iii)(d) of $F_{k,\Lambda'}$. Then the transfer maps the subfunctor $\bar{F}_{k,\Lambda'} \subset F_{k,\Lambda'}$ to $\bar{F}_{k,\Lambda} \cdot ? \subset F_{k,\Lambda} \cdot ?$.

Proof: See section IV.

Actually, it may be just as well to unveil the idea behind II.4(b) and proposition II.6 . Let (C,λ) be the usual object in $\tilde{\mathcal{C}}_{\Lambda}$, and write I for the cellular chain complex of the unit interval; also denote by $\{0\}$ and $\{1\}$ the two subcomplexes of I which deserve that appellation.

$C\otimes_{\mathbb{Z}} I$, or $C\otimes I$ for short, is then again a complex of f.g. free Λ-modules (Λ acts on it according to the rule $\alpha(c\otimes i) = \alpha(c)\otimes i$).

The 'projection' $pr: C\otimes I \longrightarrow C$ defines a cycle $\lambda\otimes I$ in $Q(C\otimes I)_0$, namely the pullback of $\lambda \in Q(C)_0$. Then $(C\otimes I, \lambda\otimes I)$ is a new object in $\tilde{\mathcal{C}}_\Lambda$, and we have two morphisms in $\tilde{\mathcal{C}}_\Lambda$,

$$i_1 : (C,\lambda) \cong (C\otimes\{0\}, \lambda\otimes\{0\}) \hookrightarrow (C\otimes I, \lambda\otimes I)$$

and $$i_2 : (C,\lambda) \cong (C\otimes\{1\}, \lambda\otimes\{1\}) \hookrightarrow (C\otimes I, \lambda\otimes I) .$$

II.7 PROPOSITION (addendum to II.4):
$\mathrm{Im}(\phi_{(C,\lambda)}) \subset F_k((C,\lambda))$ equals the subgroup

$$\{z \in F_k((C,\lambda)) \mid i_{1*}(z) = i_{2*}(z)\} .$$

Together with II.4(a) , this clearly implies II.4(b) and II.6 .

III. THE GROUPS $S_{2k}(\cdots)$

III.1 PROPOSITION: Let $\gamma : E \longrightarrow B$ be a real (stable) vector bundle over the connected CW-space B (which is assumed to have a finite number of cells in each dimension only), and (p, ω) a point in the orientation cover B^{or} (so $p \in B$, and ω is an orientation of the fibre at p).
Then γ determines an object in $\mathcal{H}\tilde{\mathcal{E}}_\Lambda$, up to unique isomorphism (in $\mathcal{H}\tilde{\mathcal{E}}_\Lambda$). ($\Lambda = \mathbb{Z}[\pi_1(B,p)]$.)

Sketch proof: First of all, notice that a bundle $\gamma : E \longrightarrow B$ is essentially as good as a map $f : B \longrightarrow BO$. Anyway - bundles seem more convenient at the moment than maps to BO. $\gamma : E \longrightarrow B$ determines a Thom spectrum; call the associated manifolds γ-manifolds. Thus a γ-manifold is a manifold together with a map (not a homotopy class) $c : M \longrightarrow B$ and a stable isomorphism of $c^*(\gamma)$ with the normal bundle ν_M.
Right-ho. It is possible to construct a sequence of γ-manifolds (with boundary) denoted by P_n^{2n} (the <u>super</u>scripts indicate the dimension) such that:

(i) Each P_n 'mimics' the n-skeleton of the CW-complex B. (So there is given a simple homotopy equivalence $P_n \simeq$ n-skeleton of B; more precisely, P_n has one j-handle for each j-cell of B, $j \leq n$, and the composite $P_n \simeq$ n-skeleton of $B \hookrightarrow B$ is just the classifying map for the normal bundle.)

(ii) $P_n \times I \subset \partial P_{n+1}$ (I = unit interval); and the composite inclusion $P_n \times I \hookrightarrow \partial P_{n+1} \hookrightarrow P_{n+1}$ 'mimics' the inclusion n-skeleton of B $\hookrightarrow$ (n+1)-skeleton of B .

(Such a sequence (P_n) will be called a 'γ-fattening of B '.)

The relative homology group $H_n(P_n, P_{n-1} \times I)$ (coefficients $\Lambda = \mathbb{Z}[\pi_1(B,p)]$) is then canonically isomorphic to $C(\tilde{B})_n$. ($C(\tilde{B})$ is the cellular chain complex of the universal covering space of (B,p) .) On the other hand, $H_n(P_n, P_{n-1} \times I)$ is also equipped with a sesquilinear form, namely the adjoint of the composite homomorphism

$$H_n(P_n, P_{n-1} \times I) \xrightarrow{\text{sliding}} H_n(P_n, \partial P_n - \mathrm{Int}(P_{n-1} \times I)) \longrightarrow \cdots$$

$$\cdots \xrightarrow{\text{Poincare duality}} H^n(P_n, P_{n-1} \times I) \cong \text{dual space of } H_n(P_n, P_{n-1} \times I) \ .$$

(The homomorphism 'sliding' is defined so as to make the diagram

$$\begin{tikzcd}
H_n(P_n, P_{n-1} \times I) \arrow[drr, "\text{sliding}"] & & \\
H_n(P_n, P_{n-1} \times \{1\}) \arrow[u, "\cong"] \arrow[rr, "(\text{inclusion})_*"'] & & H_n(P_n, \partial P_n - \mathrm{Int}(P_{n-1} \times I))
\end{tikzcd}$$

commutative.

Note that the Poincare duality isomorphism is well defined only because we have specified a preferred orientation of the fibre of γ at p ; if the other orientation is used, the sesquilinear form changes sign.)

Hence we have a sesquilinear form on each module $C(\tilde{B})_n$; the collection of these gives an element λ in $Q(C(\tilde{B}))_0$. (Q is the functor described in II.1.) Inspection shows that λ belongs to the kernel of ∂_0 ; so $(C(\tilde{B}), \lambda)$ is an object of $\tilde{\mathcal{C}}_\Lambda$, or of $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$.

The proof that this is well determined up to unique isomorphism (in $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$) involves a relative version of the construction above:

III.2 LEMMA. Let $\eta: T \longrightarrow X$ be a real vector bundle as above (over a CW-space X with finitely many cells in each dimension); suppose that X' is a CW-subspace of X and $\eta': T' \longrightarrow X'$ the induced bundle over X'. Then any η'-fattening $(Q_n')_{n=0,1,\ldots}$ of X' can be extended to an η-fattening $(Q_n)_{n=0,1,\ldots}$ of X (so that $Q_n' \subset Q_n$ etc.).

Therefore if $({}_0P_n)_{n=0,1,\ldots}$ and $({}_1P_n)_{n=0,1,\ldots}$ are two distinct γ-fattenings of B (notation as in III.1 again), we may consider the disjoint union $({}_0P_n \cup {}_1P_n)_{n=0,1,\ldots}$ as a fattening of the subspace $B\times\{0,1\}$ of $B\times I$; according to the lemma, this can be extended to a $(\gamma\times\mathrm{id})$-fattening of $B\times I$ ($\gamma\times\mathrm{id}: E\times I \longrightarrow B\times I$).
All this gives rise to a diagram in $\tilde{\mathcal{C}}_\Lambda$ (or $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$)

$$(C(\widetilde{B\times\{0\}}),\ {}_0\lambda) \longrightarrow (C(\widetilde{B\times I}),\ {}_I\lambda) \longleftarrow (C(\widetilde{B\times\{1\}}),\ {}_1\lambda) .$$

The two morphisms here are isomorphisms in $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$, because the underlying chain maps are homotopy equivalences. Hence the two distinct fattenings of B determine isomorphic objects in $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$. A similar argument shows that the isomorphism is well determined, too.

III.3 COROLLARY: Any bundle over the CW-space B (with finitely many cells in each dimension, say) determines a homology class in $H_0(Q(C(\tilde{B})))$.

This is more easily digested (i.e. proved) together with

III.4 LEMMA: Let C , C' be two chain complexes in $\mathcal{C}_\Lambda$ ($\Lambda = \mathbb{Z}[\pi_1(B)]$, say). Then chain homotopic maps from C to C' induce identical homomorphisms from $H_*(Q(C'))$ to $H_*(Q(C))$.

Proof: Let $s = (s_n : C_{n-1} \longrightarrow C'_n)_{n\in\mathbb{Z}}$ be a homotopy from f to g, so that $f + \partial s + s\partial = g$. Then for any cycle λ' in $Q(C')_0$,

$$f^*(\lambda') + \delta_1(s\&\lambda') = g^*(\lambda') \qquad \text{in } Q(C)_0 ,$$

where $s\&\lambda' \in Q(C)_1$ is defined by

$$(s\&\lambda')_n(x,y) = (-1)^{n+1}\cdot T\lambda'_{n+1}(s_{n+1}(x), s_{n+1}(y))$$
$$+ \lambda'_n(g(x), \partial s_{n+1}(y)) + \lambda'_n(s_n\partial(x), f(y)) \quad .$$

Cycles in $Q(C')_1$ can be dealt with similarly; cf. V.1 .

III.5 REMARK: In III.3 , $H_0(Q(C(\tilde{B})))$ should be interpreted as a globally constant sheaf on B^{or}, as usual .

The next few pages are devoted to an alternative description of the category $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$, which will in turn yield a more lucid version of III.1 . Let $C := \cdots \longleftarrow C_{-1} \longleftarrow C_0 \longleftarrow C_1 \longleftarrow \cdots$ be a chain complex in $\mathcal{C}_\Lambda$ (Λ any ring with involution). It determines another chain complex $Q(C) := \cdots\ Q(C)_{-1} \longleftarrow Q(C)_0 \longleftarrow Q(C)_1 \longleftarrow \cdots$ of abelian groups, periodic with period two (see II.1) .

Associated with any chain complex of abelian groups, there is a simplicial abelian group (whose geometric realization 'usually' is a topological abelian group) . This construction K (sketched far below; see also [CURTIS]) is due to Dold and Kan ; it is functorial and, when restricted to the subcategory of chain complexes which are trivial in negative dimensions, gives an isomorphism of this subcategory with the category of simplicial abelian groups. For such a chain complex D , the homotopy groups of K(D) are naturally isomorphic to the homology groups of D .

In general (i.e. even if D is nontrivial in negative dimensions) ,

$K(D) = K(D^+)$, where D^+ is the positive half of D ; thus

$$D_n^+ = \begin{cases} D_n & n>0 \\ \ker[D_0 \longrightarrow D_{-1}] & n=0 \\ 0 & n<0 \end{cases} .$$

Returning to the chain complexes C and $Q(C)$, we find that the group of components of (the geometric realization of) $K(Q(C))$ is isomorphic to $H_0(Q(C))$. Proposition III.1 states, roughly, that a bundle on a CW-complex B determines a subtlety which occupies an intermediate position between a cycle in $Q(C(\tilde{B}))_0$ and the corresponding homology class in $H_0(Q(C(\tilde{B})))$. We want to come to grips with this subtlety.

III.6 DEFINITION: Let C be a chain complex in $\mathcal{C}_\Lambda$, as before. A toy bundle on C is a simplicial map (of simplicial sets)

$$t; X \longrightarrow K(Q(C))$$

such that X is simply connected (and connected) , and t is a covering map (i.e. a Kan fibration with discrete fibres).

The improved version of III.1 is

III.7 PROPOSITION. Assumptions and notation being as in III.1, the bundle γ determines a toy bundle on $C(\tilde{B})$, unique up to unique isomorphism ; here $C(\tilde{B})$ denotes the cellular chain complex of the universal covering space of (B,p). (The proof will have to wait.)

III.8 ELUCIDATION (of III.6)

(i) The cycles in $Q(C)_0$ are in 1-1-correspondence with the 0-simplices of $K(Q(C))$; hence any such cycle determines a toy bundle, given by the universal covering space associated with the corresponding 0-simplex and its component. Any toy bundle on C is isomorphic (over C) to one obtained in this manner; further, two toy bundles on C are isomorphic if and only if they determine the same element in $H_0(Q(C))$ (or the same component of $K(Q(C))$). However, the isomorphism is not unique in general; in fact, the automorphism group of any fixed toy bundle on C is canonically isomorphic to $H_1(Q(C)) \cong \pi_1(K(Q(C)))$.

(ii) Lemma: Let $f:A \longrightarrow B$ be a continuous map between CW-spaces (or a simplicial map between simplicial sets), and suppose $t:Y \longrightarrow A$ is a (Kan) fibration with discrete fibres, Y 1-connected (hence connected). Then f induces a (Kan) fibration $f_*t\colon f_*Y \longrightarrow B$ with similar properties.

Proof (of the topological version): Let f_*Y consist of homotopy classes of triples (y,ω,b) where $y \in Y$, $b \in B$, and ω is a path in B connecting $f \cdot t(y)$ with b . (The homotopies in question are allowed to vary y and ω , but not the endpoint b .) Define $f_*t\colon f_*Y \longrightarrow B$ by sending (y,ω,b) to b .

(iii) Induced toy bundles: Let $f:C \longrightarrow C'$ be a chain map (of chain complexes belonging to $\mathcal{C}_\Lambda$), and $t:X \longrightarrow K(Q(C'))$ a toy bundle on C' . There is then an induced toy bundle f^*t on C .
For $f:C \longrightarrow C'$ induces $K(Q(f))\colon K(Q(C')) \longrightarrow K(Q(C))$, and (ii) above can be applied.

(iv) Change of rings: Assume C is a chain complex in $\mathcal{C}_\Lambda$, and $j\colon \Lambda \longrightarrow \Lambda'$ is a homomorphism of rings with involution (= involutory antiautomorphism) , giving rise to a functor $-\otimes_\Lambda \Lambda'\colon \mathcal{C}_\Lambda \longrightarrow \mathcal{C}_{\Lambda'}$.
Any toy bundle t on C gives another toy bundle $t \otimes_\Lambda \Lambda'$ on $C \otimes_\Lambda \Lambda'$.

Proof: Tensoring with Λ' gives a chain map $Q(C) \longrightarrow Q(C \otimes_\Lambda \Lambda')$, hence $K(Q(C)) \longrightarrow K(Q(C \otimes_\Lambda \Lambda'))$, and (ii) can be applied.

(v) Morphisms of complexes with toy bundles:
Let t be a toy bundle on C, t' a toy bundle on C' (C, C' in $\mathcal{C}_\Lambda$). A 'toy bundle map' from (C,t) to (C',t') is a pair (f,s), where $f: C \longrightarrow C'$ is a chain map and $s: t \cong f^* t'$ is an isomorphism of toy bundles over C.

(vi) Homotopies (notation as in (v)):
Two toy bundle maps (f_0,s_0) and (f_1,s_1) from (C,t) to (C',t') are 'homotopic' if there exists a toy bundle map (f_I,s_I) from $(C \otimes I, t \otimes I)$ to (C',t') which restricts to (f_0,s_0) and (f_1,s_1) on the respective 'boundaries'.
(Here $I = \cdots \quad 0 \longleftarrow 0 \longleftarrow Z \oplus Z \xleftarrow{(id,-id)} Z \longleftarrow 0 \longleftarrow 0 \quad \cdots$ is the cellular chain complex of the unit interval, and $C \otimes_Z I$ is considered as a chain complex of Λ-modules via the given action on C, i.e. $\alpha \cdot (c \otimes i) := \alpha \cdot c \otimes i$. The toy bundle $t \otimes I$ is induced by the projection $C \otimes I \longrightarrow C$.)

(vii) Homotopy lifting property: Suppose $(f_0,s_0): (C,t) \longrightarrow (C',t')$ is a toy bundle map, and $f_I: C \otimes I \longrightarrow C'$ is a homotopy from f_0 to another chain map f_1. Then f_I has a unique lifting to a toy bundle homotopy $(f_I,s_I): (C \otimes I, t \otimes I) \longrightarrow (C',t')$ 'starting' with (f_0,s_0).

PROOF of III.7 (notation as in III.1): First it is necessary to restrict the notion of 'fattening' slightly so as to ensure the γ-fattenings of B form a set. (For instance, all the handles involved might be required to be products of cells of B with standard disks.) Next it is necessary to observe that any γ-fattening of B determines a toy bundle on $C(\tilde{B}) = C(\tilde{B}, p)$, and that the toy bundles so obtained are all canonically isomorphic. Indeed, the proof of III.1 gives a cycle in $Q(C(\tilde{B}))_0$, i.e. a point (or 0-simplex) in $K(Q(C(\tilde{B})))$, and therefore a toy bundle by III.8(i) . Further, if two γ-fattenings $\mathcal{F}_1$, $\mathcal{F}_2$ determine 0-simplices x_1 , x_2 in $K(Q(C(\tilde{B})))$, the argument following III.2 gives a preferred homotopy class of paths connecting x_1 and x_2 ; but this is precisely the same as an isomorphism of the associated toy bundles . Hence a 'choice-free' toy bundle on $C(\tilde{B})$ can be obtained by identifying all these toy bundles, one for each fattening.

To see more clearly that III.7 is a reformulation of III.1 , let us look at the category $\mathcal{D}$ whose objects are pairs (C,t) with C a chain complex in $\mathcal{C}_\Lambda$ and t a toy bundle on C , and whose morphisms are <u>homotopy</u> classes of toy bundle maps as in III.8(vi) . The functor $\tilde{\mathcal{C}}_\Lambda \longrightarrow \mathcal{D}$ given by III.8(i) has the property of mapping any morphism in $\tilde{\mathcal{C}}_\Lambda$ whose underlying chain map is a homotopy equivalence to an isomorphism in $\mathcal{D}$. Hence, by the universal property of the canonical functor $\tilde{\mathcal{C}}_\Lambda \longrightarrow \mathcal{H}\tilde{\mathcal{C}}_\Lambda$, there is a unique functor $J\colon \mathcal{H}\tilde{\mathcal{C}}_\Lambda \longrightarrow \mathcal{D}$ making the diagram

$$\begin{tikzcd}
 & \mathcal{H}\tilde{\mathcal{C}}_\Lambda \arrow[d] \\
\tilde{\mathcal{C}}_\Lambda \arrow[ur] \arrow[dr] & \\
 & \mathcal{D}
\end{tikzcd}$$

commutative.

III.9 PROPOSITION: $J: \widetilde{\mathcal{HE}}_\Lambda \longrightarrow \mathcal{D}$ is an equivalence of categories (precisely, J embeds $\widetilde{\mathcal{HE}}_\Lambda$ as a full subcategory, and every object in $\mathcal{D}$ is isomorphic to one in $\mathrm{im}(J)$).

Proof: See section V.

To end this topic, here comes the Dold-Kan construction (notation is as in [CURTIS], but the construction itself is slightly different). Each simplicial abelian group A gives a chain complex NA with

$$NA_q = \bigcap_{i \neq 0} \ker d_i \qquad \text{(the } d_i \text{ being the face operators on } A_q \text{)}$$

$$\text{and} \qquad \partial_q = d_0 \qquad \text{(restricted to } NA_q\text{)}.$$

The construction K below converts chain complexes into simplicial abelian groups, and is inverse to N when restricted to non-negatively graded chain complexes: Let $C := \cdots - C_{-1} \longleftarrow C_0 \longleftarrow C_1 \longleftarrow \cdots$ be (any) chain complex.

Define KC_q to consist of all functions f which to each subset S of $\{0,1,\cdots,q\}$ associate an element in $C_{|S|-1}$ and satisfy the equation

$$f(S) = \sum_{0 \leq i < S} (-1)^i \cdot f(d_i S)$$

(where $d_0 S$ is the subset of S obtained by deleting the least element of S, $d_1 S$ is obtained by deleting the next, etc.). Face and degeneracy operators are obvious, and the claims made on p.III.4 are easy to verify.

III.10 DEFINITION (of the functor S_{2k} on 'nice' objects):
Let $\gamma: E \longrightarrow B$ be as in III.1 ; for each point (p,ω) in B^{or}, γ gives a toy bundle on $C(\widetilde{B},p)$, the cellular chain complex of the universal covering space of (B,p) ; call it $t_{(p,\omega)}$. The functor

$\bar{F}_k$ of II.3(vi) can be applied to the pair $(C(\tilde{B},p),\ t_{(p,\omega)})$, via III.9, and gives an abelian group $S_{2k}(\gamma;p,\omega)$. The groups $S_{2k}(\gamma;p,\omega)$ constitute a locally constant sheaf on B^{or}.

III.11 LEMMA: $S_{2k}(\gamma;p,\omega)$ is a globally constant sheaf on B^{or}.

We may therefore unambiguously define $S_{2k}(\gamma)$ to be the abelian group of sections of that sheaf.

Proof: The proof of III.1 showed that, for any point $(p,\omega)\in B^{or}$, every γ-fattening $\mathcal{F}$ determines a cycle $\lambda_{\mathcal{F},p,\omega}$ in $Q(C(\tilde{B},p))_0$. The functor F_k of II.3(iii) (not $\bar{F}_k$) can be applied to the pair $(C(B\ ,p),\ \lambda_{\mathcal{F},p,\omega})$ to give a possibly noncommutative group. The result is a locally constant sheaf of groups Γ on B^{or}. It suffices to prove that Γ is globally constant, because $S_{2k}(\gamma)$ is a subsheaf of Γ, according to proposition II.4.

Proof of this (sketchy): Fix attention on the point $(p,\omega)\in B^{or}$ and on the fattening $\mathcal{F}$ again. The pair $(C(\tilde{B},p),\ \lambda_{\mathcal{F},p,\omega})$ can be considered as an object of $\tilde{\mathcal{C}}_\Lambda$, with $\Lambda = \mathbb{Z}[\pi_1(B,p)]$; on the other hand, from the way it was constructed, it appears as just one stalk of a 'locally constant sheaf' of similar pairs.

Now it is important to realize

(a) that every object of $\tilde{\mathcal{C}}_\Lambda$ can be thought of as a sheaf on B^{or} in this manner,

and (b) that, instead of applying F_k fibrewise and obtaining sheaves like Γ above, we can reformulate the construction of F_k in these terms. (That is, we may redefine F_k as a functor from the category of sheaves as in (a) to the category of groups; here it must be remembered that the sheaf L_{2k} on B^{or} is globally constant, which is important e.g. for the translation of II.3(ii)).

This new definition of F_k amounts to a trivialization of Γ and its siblings.

The definition of the functor S_{2k} on morphisms (i.e. bundle maps) is clear from II.3(iv) , proposition II.4 , and from

III.12 PROPOSITION: Bundle maps induce toy bundle maps on the chain complex level; similarly, homotopies of bundle maps induce toy homotopies. To be precise, let $\gamma_1 : E_1 \longrightarrow B_1$, $\gamma_2 : E_2 \longrightarrow B_2$ be as in III.1 , and let

$$\begin{tikzcd} E_1 \arrow[r, "\bar{f}"] \arrow[d, "\gamma_1"'] & E_2 \arrow[d, "\gamma_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

be a bundle map (= pullback square) ; suppose f is cellular.
Fix $(p,\omega) \in B_1^{or}$ and put $\Lambda_1 = \mathbb{Z}[\pi_1(B_1,p)]$, $\Lambda_2 = \mathbb{Z}[\pi_1(B_2,f(p))]$.

Denote by $C(f): C(\tilde{B}_1,p) \otimes_{\Lambda_1} \Lambda_2 \longrightarrow C(\tilde{B}_2,f(p))$
the map of chain complexes induced by f .
If $t_{p,\omega}$ is the toy bundle on $C(B_1,p)$ determined by γ_1 ,
and $t_{f(p),f(\omega)}$ the toy bundle on $C(\tilde{B}_2,f(p))$ determined by γ_2 ,

THEN

$C(f)^* t_{f(p),f(\omega)}$ and $t_{p,\omega} \otimes_{\Lambda_1} \Lambda_2$ are isomorphic toy bundles on $C(B_1,p) \otimes_{\Lambda_1} \Lambda_2$, by a preferred isomorphism.

(Proof: Exercise again, with the following hints: Assume $f: B_1 \longrightarrow B_2$ is the inclusion of a CW-subcomplex, and use III.2 .)

Finally, we can get rid of any finiteness assumptions on the base space B by using a direct limit procedure; that is, for general $\gamma : E \longrightarrow B$, we define

$$S_{2k}(\gamma) := \varinjlim S_{2k}(\gamma/_{B_i})$$

where the limit is taken over all subspaces B_i of B with only finitely many cells in each dimension. The same procedure works for morphisms. We can, moreover, do without cell decompositions altogether by just assuming B to be a (reasonable) topological space, and then replacing it by its singular semisimplicial set , etc. .

III.13 PROPOSITION: The functor S_{2k} is invariant under homotopies of bundle maps .

(Proof:clear.)

III.14 DEFINITION of the transformation λ in section 0 :
We have defined $S_{2k}(\gamma)$ as $\overline{F}_k((C,\lambda))$ for a certain pair (C,λ) , or rather a sheaf of such pairs; hence observation II.5 can be taken as a definition of λ .

There remains the more difficult task of defining the transformation σ from section 0 (recall that σ is supposed to give a homomorphism from $\pi_{2k}(M(\gamma))$ to $S_{2k}(\gamma)$ for every $\gamma: E \longrightarrow B$). This requires a (partial) geometric description of $S_{2k}(\gamma)$.

III.15 PROPOSITION: Let $\gamma : E \longrightarrow B$ be as in III.1 , (p,ω) a point in B^{or} , and $(P_n^{2n})_{n=0,1,\ldots}$ a γ-fattening of B (so each P_n is a compact γ-manifold of dimension $2n$, etc.). Let $(C(\tilde{B}),\lambda)$ be the object in $\tilde{\mathcal{E}}_\Lambda$ determined by these data (as in the proof of III.1 ; $\Lambda = \mathbb{Z}[\pi_1(B,p)]$).

Then, for $k>3$, $F_k((C(\tilde{B}),\lambda))$ is 'naturally' isomorphic to $G(P_{k-1})$ (the latter group is going to be defined below, in terms of the manifold P_{k-1} , in a geometric way). More precisely, there is a 'natural' homomorphism

$$G(P_{k-1}) \longrightarrow F_k((C(\tilde{B}),\lambda)) ,$$

defined for $k>0$, which is an isomorphism for $k>3$, and injective for $k=3$.

Thus, whatever $G(P_{k-1})$ is, it contains $S_{2k}(\gamma)$ as a subgroup (since $S_{2k}(\gamma) \cong \bar{F}_k((C(\tilde{B}),\lambda)) \subset F_k((C(\tilde{B}),\lambda))$).

III.16 DEFINITION of $G(-)$:

Let P^{n-2} be any compact γ-manifold with boundary (in particular, P may be P_{k-1} , $n=2k$). The elements of $G(P)$ are to be equivalence classes represented by pairs (N^n,j) ; here N is a compact γ-manifold and $j: P\times I \hookrightarrow \partial N$ is a γ-embedding (see explanation below, if necessary) of codimension zero, giving an open book decomposition for ∂N with page P . (In other words, it is required that $\partial N - \mathrm{int}(j(P\times I))$, which can be considered as a (co-)bordism modulo boundary from $P\times\{0\}$ to $P\times\{1\}$, be a product cobordism; see [QUINN] .)

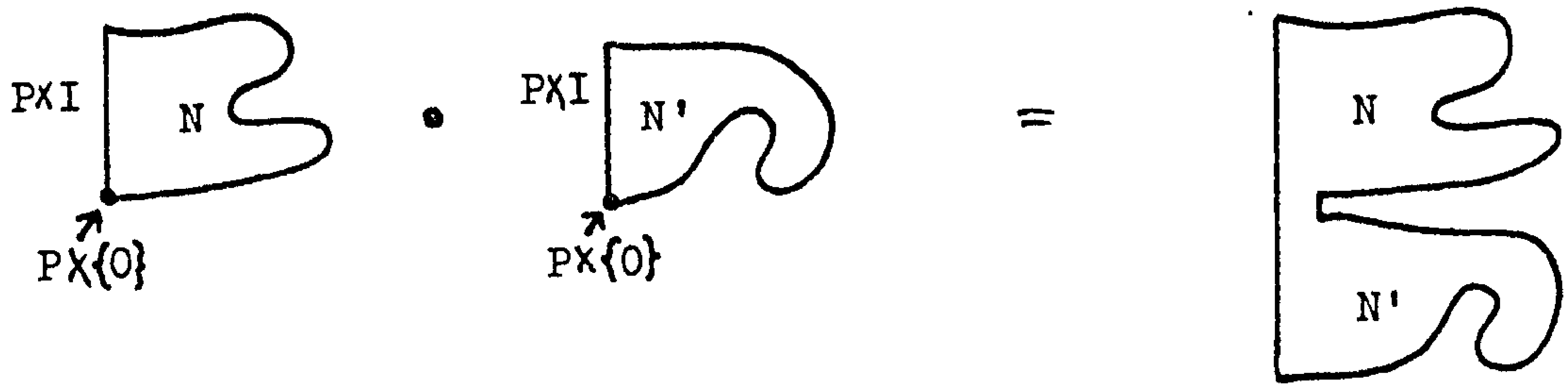
Two such pairs (N_1,j_1) and (N_2,j_2) are equivalent if there exists a γ-diffeomorphism $\alpha : \partial N_1 \longrightarrow \partial N_2$ such that the diagram of

γ-embeddings

$$\begin{tikzcd} \partial N_1 \arrow[rr] & & \partial N_2 \\ & P\times I \arrow[ul, "j_1"] \arrow[ur, "j_2"'] & \end{tikzcd}$$

is strictly commutative, and so that the closed γ-manifold $(N_1 \cup_{\alpha} -N_2)$ represents $0 \in \pi_{2k}(M(\gamma))$.

(Multiplication in $G(P)$ is indicated in the picture:



In particular, the neutral element is represented by $(P\times I\times I,\ j: P\times I \cong P\times I\times\{0\} \hookrightarrow P\times I\times I)$, and if $x \in G(P)$ is represented by (N^n, j) , then x^{-1} is represented by $(-N, \varrho)$, where ϱ is j upside down, of course.)

G behaves functorially: any γ-embedding $e: P_1 \longrightarrow P_2$ of γ-manifolds (mapping P_1 into the interior of P_2 , say) induces a homomorphism $G(e): G(P_1) \longrightarrow G(P_2)$.

(If $x \in G(P_1)$ is represented by (N^n, j) , then $G(e)(x)$ is represented by $(P_2\times I\times I \cup_{P_1\times I} N\ ,\ i: P_2\times I \cong P_2\times I\times\{0\} \hookrightarrow P_2\times I\times I \cup N)$.

Here the union $P_2\times I\times I \cup_{P_1\times I} N$ is obtained by glueing together the two copies $e(P_1)\times I\times\{1\}$ and $j(P_1\times I)$ of $P_1\times I$.)

EXPLANATION (of the expression 'γ-embedding'): Let P_1, P_2 be two γ-manifolds. Then we have pullback squares

$$\begin{tikzcd}
E(\gamma_i) \arrow[r] \arrow[d] & E \arrow[d] \\
P_i \arrow[r] & B
\end{tikzcd}$$

for $i = 1,2$, $E(\gamma_i)$ being the total space of the normal bundle of P_i (and γ_i is sufficiently well determined to be called 'the', since the classifying map $P_i \longrightarrow BO$ is well determined up to an infinity of higher homotopies). Any embedding (in the usual sense) $P_1 \longrightarrow P_2$ defines a second 'γ-structure' on P_1, namely the composite

$$\begin{tikzcd}
E(\gamma_1) \arrow[r, "\text{differential}"] \arrow[d] & E(\gamma_2) \arrow[r] \arrow[d] & E \arrow[d] \\
P_1 \arrow[r] & P_2 \arrow[r] & B
\end{tikzcd}.$$

For our purposes, a γ-embedding $P_1 \hookrightarrow P_2$ is simply an embedding in the usual sense, plus a preferred homotopy class of homotopies between the two γ-structures on P_1.

The proof of III.15 will occupy the last part of section IV. Here we are only concerned with the description of the transformation

$\delta: \pi_{2k}(M(B,f)) \longrightarrow S_{2k}(B,f)$, or $\delta: \pi_{2k}(M(\gamma)) \longrightarrow S_{2k}(\gamma)$,

taking III.15 for granted.

In the notation of III.15, put $V := P_{k-1} \times I$, and return to section I, especially I.6. V satisfies the conditions in I.6(i). What is more, $G(P_{k-1})$ is contained (as a set) in the set $\mathcal{S}(V)$ defined in I.6(ii) ; indeed, every element in $G(P_{k-1})$ is represented by something which can be regarded as a cobordism (modulo boundary) from $V = P_{k-1} \times I$ to another manifold.

In I.6(iii) , a map $\sigma_V \colon \pi_{2k}(M(\gamma)) \longrightarrow \mathcal{S}(V)$ was defined. This is easily seen to factor:

$$\pi_{2k}(M(\gamma)) \longrightarrow G(P_{k-1}) \longrightarrow \mathcal{S}(V) \quad ,$$

giving a homomorphism

$$\pi_{2k}(M(\gamma)) \longrightarrow G(P_{k-1}) \longrightarrow F_k((C(\tilde{B}), \lambda))$$

(notation as in III.15).

It will be shown in section IV (among other things) that this homomorphism factors again:

$$\pi_{2k}(M(\gamma)) \longrightarrow S_{2k}(\gamma) = \bar{F}_k(\cdots) \hookrightarrow F_k(\cdots)$$

(the homomorphism on the right is injective by II.4(a)) , and this defines σ .

Note that the same procedure can be employed to give what turns out (also in section IV) to be a second definition of the natural transformation $\lambda : L_{2k}(\cdots) \longrightarrow S_{2k}(\cdots)$. Granting that, I.4 is a corollary of III.15 .

IV.PROOFS

We begin with a proof of III.15 ; the presentation will be unashamedly horrible .

First, a new description of the functor F_{2k} from II.3(ii) is required. To prepare for it, here is a reminder on quadratic forms .

IV.1 REMINDER: Let C be a stably free f.g. Λ-module, with a preferred equivalence class of bases (Λ a ring with involution). Then, according to the quaint view, a $(-1)^k$-quadratic (or -hermitian) form on C is a pair (λ, μ) where λ is a sesquilinear form on C (see beginning of section II) which satisfies the symmetry condition

$\lambda(x,y) = (-1)^k \cdot \lambda(y,x)^-$ for $x,y \in C$, and where

$\mu : C \longrightarrow \Lambda/\{\gamma-(-1)^k\cdot\bar{\gamma} \mid \gamma\in\Lambda\}$ satisiies the 'quadratic law'

$$\mu(x+y) = \mu(x) + \mu(y) + \lambda(x,y)$$

and a few conditions on top of that; see [WALL 1, Thm.5.2] .

The more elegant approach ([WALL 1, ch. 17G] or [WALL 2]) is to say that a $(-1)^k$-quadratic form on C is an element in

$$\operatorname{coker}(\ 1 - (-1)^k T\ ) \quad ,$$

where $T: \mathrm{Sel}(C) \longrightarrow \mathrm{Sel}(C)$ is the usual involution on the space of sesquilinear forms (and 1 is the identity on Sel(C)).

To see the equivalence of the two definitions, let $\bar{q} \in \mathrm{Sel}(C)$ represent the coset $q \in \operatorname{coker}(\ 1-(-1)^k T\ )$, and put

$q^? := \bar{q} + (-1)^k T(\bar{q})$; also define $\mu : C \longrightarrow \Lambda/ \ldots$

by $\mu(x) := \text{coset of } \bar{q}(x,x)$ for $x \in C$.

Then $(q^?, \mu)$ is a $(-1)^k$-quadratic form according to the first definition, and depends only on the coset q of $\bar{q}$.

Now let (C,λ) be an object in $\tilde{\mathcal{C}}_\Lambda$ (see II.2) with underlying chain complex $C : \cdots \longrightarrow C_{-1} \longleftarrow C_0 \longleftarrow C_1 \longleftarrow \cdots$.

IV.2 DEFINITION. A 'k-extension' of (C,λ) is a quintuple (D,i,f,b,q) , where

(1) D is a Λ-module, and the homomorphisms i, f form part of a commutative diagram

$$\begin{tikzcd} & & & D \arrow[dll, "f"'] & \\ C_{k-1} & C_k \arrow[l] & & C_{k+1} \arrow[ll, "\partial_{k+1}"] \arrow[u, "i"'] & C_{k+2} \arrow[l] & \cdots \arrow[l] \end{tikzcd} ;$$

(2) the homomorphism i is injective, $D/\mathrm{im}(i)$ is stably free and finitely generated, and b is an equivalence class of bases for $D/\mathrm{im}(i)$;

(3) q is a $(-1)^k$-quadratic form on D , satisfying

$$i^*(q) = \text{coset of } \lambda_{k+1} \text{ in } \mathrm{coker}(\ 1-(-1)^kT\ )$$

(so $q^?$ restricts to the symmetrization of λ_{k+1} on C_{k+1} etc.)

and
$$q^?(x,y) = \lambda_k(f(x),f(y)) \qquad \text{for} \quad x \in \mathrm{im}(i) \subset D\ ,\ y \in D\ .$$

If the following condition is also satisfied, the k-extension will be called 'simple nondegenerate' :

(4) the sesquilinear form $q^? - f^*(\lambda_k)$ which, by condition (3) , may be thought of as defined on $D/\mathrm{im}(i)$, is simple nondegenerate (on $D/\mathrm{im}(i)$).

IV.3 ILLUMINATION: (notation as in IV.2): Let $\bar{q} \in \mathrm{Sel}(D)$ be a sesquilinear form representing q (as in IV.1) and restricting to λ_{k+1} on C_{k+1} . (Such a $\bar{q}$ is easy to construct.) Then a new object in $\tilde{\mathcal{C}}_\Lambda$ can be manufactured (lower row in the following commutative diagram) , containing (C,λ) :

$$\begin{tikzcd}
 & & C_k \arrow[dl] \arrow[dd, "j_k"] & C_{k+1} \arrow[l] \arrow[dd, "j_{k+1}"] & & \\
C_{k-2} & C_{k-1} \arrow[l] & & & C_{k+2} \arrow[ul] \arrow[dl] & \cdots \arrow[l] \\
 & & C'_k \arrow[ul, "\partial'_k"] & C'_{k+1} \arrow[l, "\partial'_{k+1}"] & &
\end{tikzcd} .$$

Here

$$\begin{aligned}
C'_{k+1} &= D \ ; \\
C'_k &= C_k \oplus D/_{\mathrm{im}(i)} \ ; \\
\partial'_{k+1} &= (f,\text{projection}) \ ; \\
\partial'_k &= \partial_k \oplus -\partial_k \cdot f \qquad (\ \partial_k \cdot f \ \text{ is really defined on } D, \\
&\qquad\qquad\qquad\qquad \text{but vanishes on } \mathrm{im}(i)\ ); \\
j_k &= \text{inclusion of left summand} \ ; \\
j_{k+1} &= i \ ; \\
\lambda'_{k+1} &= \bar{q} \ ;
\end{aligned}$$

and λ'_k is given by

$$\lambda'_k(x,y) = \begin{cases} q^{?}(x,y) - (-1)^k \lambda_k(f(y),f(x))^{-} & \text{for } x,y \in D/_{\mathrm{im}(i)} \\ & \text{(see remark IV.4);} \\ 0 & \text{if } x \in C_k,\ y \in D/_{\mathrm{im}(i)} \ ; \\ \lambda_{k-1}(-\partial_k \cdot f(x), \partial_k(y)) & \text{if } x \in D/_{\mathrm{im}(i)},\ y \in C_k \ ; \\ \lambda_k(x,y) & \text{if } x,y \in C_k \ . \end{cases}$$

Notice that, in the definition of λ'_k , only the second line really matters (which says that C'_k splits as a left orthogonal sum); the others are consequences .

The procedure works backwards, of course; a diagram as above (whatever that means) gives a k-extension. Two things are important to remember:

(1) a splitting for j_k is needed; and (2) only the class of λ'_{k+1} in $\operatorname{coker}(1-(-1)^k T)$ matters (see IV.1) .

IV.4 REMARK: In memorizing this definition, it may help to consider only the free and based module $D/_{\mathrm{im}(i)}$, the sesquilinear form
$\eta := q^? - (-1)^k T(f^*(\lambda_k)) = (-1)^k T(q^? - f^*(\lambda_k))$ on it (expl. below) ,
and the map $g := -\partial_k \cdot f : D/_{\mathrm{im}(i)} \longrightarrow C_{k-1}$ as essential, and everything else as ornament.
(Explanation: T is the usual involution on the space of sesquilinear forms. The definition of η makes sense because of IV.2(3) ; further, η is simple nondegenerate precisely if the whole k-extension is (IV.2(4)). IV.3 explains why η is important: definition of λ'_k , top line .
Notice that g, η and λ_{k-1} are related: $g^*(\lambda_{k-1}) = \eta - (-1)^k T(\eta)$.)

Two distinct k-extensions of (C,λ) can be multiplied :

IV.5 PROPOSITION. Let $E_1 = (D_1,i_1,f_1,b_1,q_1)$ and $E_2 = (D_2,i_2,f_2,b_2,q_2)$ be two k-extensions of (C,λ) . Then there exists a unique k-extension $E_1 \cdot E_2 = (D,i,f,b,q)$ such that

(1) D is the pushout of
$$\begin{tikzcd} & D_1 \\ C_{k+1} \arrow[ur] \arrow[dr] & \\ & D_2 \end{tikzcd}$$
;

(2) i, f and b are what you expect;

(3) q restricts to q_1 and q_2 on D_1 , D_2 respectively (both modules may be considered as submodules of D);

(4) $q^?(y,x) = \lambda_k(f(y),f(x))$ for $x \in D_1$, $y \in D_2$.

Proof: Straightforward, with the 'quaint' definition of 'quadratic form' .

Looking at $E_1 \cdot E_2$ as suggested in IV.4 gives

$$D/\mathrm{im}(i) \;=\; D_1/\mathrm{im}(i_1) \oplus D_2/\mathrm{im}(i_2) \quad , \qquad g = g_1 \oplus g_2 \quad ,$$

and η is determined by the compatibility condition

$g^*(\lambda_{k-1}) = \eta - (-1)^k T(\eta)$ and a 'left-orthogonality' assumption, i.e.

$$\eta(x,y) = \begin{cases} \eta_1(x,y) & \text{for } x,y \in D_1/\mathrm{im}(i_1) \\ \eta_2(x,y) & x,y \in D_2/\mathrm{im}(i_2) \\ 0 & x \in D_1/\mathrm{im}(i_1) \ , \ y \in D_2/\mathrm{im}(i_2) \end{cases} .$$

IV.6 CONSTRUCTION: Let $f:(C,\lambda) \longrightarrow (C',\lambda')$ be a morphism in $\tilde{\mathcal{C}}_\Lambda$, and $E = (D,i,f,b,q)$ a k-extension of (C,λ). E and f determine an induced k-extension of (C',λ'), namely $f_*E = (D',i',f',b',q')$ such that e.g.

D' is the pushout of the diagram
$$\begin{tikzcd} & D \\ C_{k+1} \arrow[ur, "i"] \arrow[dr, "f"'] & \\ & C'_{k+1} \end{tikzcd} .$$

A similar construction works when a change of rings is involved; i.e. when (C,λ) is in $\tilde{\mathcal{C}}_\Lambda$, (C',λ') in $\tilde{\mathcal{C}}_{\Lambda'}$, $j:\Lambda \longrightarrow \Lambda'$ is a homomorphism of rings with involution, and f is a morphism in $\tilde{\mathcal{C}}_{\Lambda'}$ from $(C,\lambda) \otimes_\Lambda \Lambda'$ to (C',λ').

IV.7 REMARK: IV.5 and IV.6 show that a covariant functor can be defined on $\tilde{\mathcal{C}}_\Lambda$ by associating with every (C,λ) in $\tilde{\mathcal{C}}_\Lambda$ the monoid of isomorphism classes of k-extensions of (C,λ) (for a fixed k).

IV.8 DEFINITION: Say that a k-extension $E = (D,i,f,b,q)$ of (C,λ) is <u>hyperbolic</u> if there exists a submodule $K \subset D$, f.g. free and based in its own right, such that $q/_K = 0$, $f/_K = 0$, and the composite $u: K \hookrightarrow D \longrightarrow D/_{im(i)}$ makes K a 'lagrangian' for the sesquilinear form η on $D/_{im(i)}$.

(The assumptions on K imply that $v \cdot u$ 0 in the sequence

$$(*) \qquad K \xrightarrow{u} D/_{im(i)} \xrightarrow{v} K^* \quad ,$$

in which v denotes the composite

$$D/_{im(i)} \xrightarrow[\text{adjoint of } \eta]{} (D/_{im(i)})^* \xrightarrow[u^*]{} K^* .$$

If $(*)$ is short exact and based (see [MILNOR 2]) , we call K a lagrangian.)

Thus if E is a hyperbolic k-extension, it must be simple nondegenerate.

Hyperbolic extensions, or their isomorphism classes, form a submonoid of the monoid in IV.7 ; what is more, a central submonoid .

IV.9 PROPOSITION: Let $E_1 = (D_1,i_1,f_1,b_1,q_1)$, $E_2 = (D_2,i_2,f_2,b_2,q_2)$ be two k-extensions of (C,λ) , and assume that E_1 is hyperbolic . Then the k-extensions $E_1 \cdot E_2$ and $E_2 \cdot E_1$ are isomorphic .

PROOF: Observe first that IV.4 has some sort of converse:
If $E = (D,i,f,b,q)$ is a k-extension of (C,λ) , and two submodules $D_1 \subset D$, $D_2 \subset D$ are given so that $D_1 + D_2$ D , $D_1 \cap D_2 = im(i)$, and if both $D_1/_{im(i)}$ and $D_2/_{im(i)}$ are f.g. free with preferred bases b_1 , b_2 respectively such that $b_1 + b_2 = b$, THEN the decomposition $D = D_1 + D_2$ gives a product decomposition $E \cong E_1 \cdot E_2$ provided condition IV.5(4) is satisfied .
This observation can be applied to the k-extension $E = E_2 \cdot E_1$; then

D is the pushout of $C_{k+1} \rightrightarrows \begin{matrix} D_2 \\ D_1 \end{matrix}$, and we must find a suitable decomposition

$$D = D_1' + D_2'$$

showing that $E = E_1 \cdot E_2$.

(The obvious decomposition $D_1' = D_1$, $D_2' = D_2$ will not do because then IV.5(4) is only satisfied the wrong way round.)

Put $D_1' = D_1$, and $D_2 := \{y + h(y) \mid y \in D_2\}$,

where $h: D_2 \longrightarrow K \hookrightarrow D_1$ D is a 'suitable' homomorphism of Λ-modules vanishing on $\mathrm{im}(i) \cong C_{k+1}$.

(Here K is a 'lagrangian' as in IV.8 .)

Whatever 'suitable' means, it is clear that the k-extension of (C, λ) obtained by restricting $E = E_2 \cdot E_1 = (D, i, f, b, q)$ to D_2' is isomorphic to $E_2 = (D_2, i_2, f_2, b_2, q_2)$; the map $D_2 \longrightarrow D_2'$ sending y to $y + h(y)$ is an isomorphism (this uses IV.5(4)).

To define h , note that the decomposition $D = D_1' + D_2'$ will give a product decomposition $E = E_1 \cdot E_2$ if and only if IV.5(4) is satisfied, i.e.

$$(1) \quad q^?(z,x) = \lambda_k(f(z), f(x)) \qquad \text{for } x \in D_1' = D_1 , \ z \in D_2' .$$

Write $z = y + h(y)$, for some $y \in D_2$, and remember that we do have

$$q^?(x,y) = \lambda_k(f(x), f(y))$$

(since $E = E_2 \cdot E_1$) , or in other words (using the symmetry of $q^?$)

$$(2) \quad q^?(y,x) = (-1)^k \lambda_k(f(x), f(y))^- .$$

Subtracting (2) from (1) , obtain

$$\begin{aligned} q^?(h(y),x) &= \lambda_k(f(y), f(x)) - (-1)^k \lambda_k(f(x), f(y))^- \\ &= \lambda_{k-1}(\partial_k \cdot f(y), \partial_k \cdot f(x)) . \end{aligned}$$

Now there exists precisely one homomorphism h satisfying that equation. For, if we keep y fixed, the expression $\lambda_{k-1}(\partial_k \cdot f(y), \partial_k \cdot f(x))$ gives an antihomomorphism $e_y : D_1 \longrightarrow \Lambda$ which vanishes on $K \cup C_{k+1} \subset D_1$. But $D_1/K+C_{k\,1} \cong K^*$, so e_y can be considered as an element in $(K^*)^* \cong K$.

IV.10 DEFINITION: For (C,λ) in $\tilde{\mathcal{E}}_\Lambda$, put

$\mathcal{E}_k((C,\lambda)) :=$ monoid of isomorphism classes of k-extensions of (C,λ), modulo hyperbolic extensions ; and

$F_k^{new}((C,\lambda)) :=$ monoid of isomorphism classes of simple nondegenerate k-extensions of (C,λ), modulo hyperbolic extensions.

Then $F_k^{new}((C,\lambda))$ is a submonoid of $\mathcal{E}_k((C,\lambda))$; both F_k^{new} and $\mathcal{E}_k$ are functors on $\tilde{\mathcal{E}}_\Lambda$.

IV.11 PROPOSITION: $F_k^{new}((C,\lambda))$ is a group.

Proof: If $E = (D,i,f,b,q)$ is a simple nondegenerate k-extension of (C,λ), put $E^{-1} := (D,i,f,b,q^{-1})$,

with $\quad q^{-1} := [f^*(\lambda_k)] - q$.

(Remember that q is an element of $\mathrm{coker}(1-(-1)^k T)$; so is $[f^*(\lambda_k)]$, the coset of $f^*(\lambda_k)$.)

Then E^{-1} is again a simple nondegenerate k-extension of (C,λ).

Let $N \subset D$ be a complement of $\mathrm{im}(i)$ in D ; so $N \cong D_{/\mathrm{im}(i)}$ is stably free and based.

Now $E \cdot E^{-1} = (\dot{D},\dot{i},\dot{f},\dot{b},\dot{q})$ where e.g. $\dot{D}$ pushout of $C_{k+1} \overset{i}{\underset{i}{\rightrightarrows}} \begin{matrix} D \\ D \end{matrix}$,

so that two inclusions $j_1 : D \hookrightarrow \dot{D}$, $j_2 : D \hookrightarrow \dot{D}$ must be distinguished.

$K := \{ j_1(n) - j_2(n) \mid n \in N \}$ is the required lagrangian for $E \cdot E^{-1}$.

(Notice that $(E^{-1})^{-1} = E$, which makes further checking unnecessary.)

The following two propositions imply III.15 when combined:

IV.12 PROPOSITION: The functors F_k^{new} and F_k on $\tilde{\mathcal{C}}_\Lambda$ are isomorphic.

IV.13 PROPOSITION: In the notation of III.15[*], $G(P_{k-1}) \cong F_k^{new}((C(\tilde{B}),\lambda))$.

PROOF of IV.12: This consists in the construction of a 'rule' R^{new} as in II.3(iii)(c) and in the verification of the conditions mentioned in II.3(iii) for the pair (F_k^{new}, R^{new}) .

Let (C,λ) be a k-regular object in $\tilde{\mathcal{C}}_\Lambda$,

$$C: = \quad \cdots \longleftarrow C_{k-1} \longleftarrow C_k \longleftarrow C_{k+1} \longleftarrow \cdots .$$

Let $E = (D,i,f,b,q)$ be the k-extension of (C,λ) given by

$D = C_{k+1} \oplus C_k$

$i =$ inclusion of left summand

$f = \partial_{k+1} \oplus \mathrm{id}$

$b =$ given basis of $C_k \cong D/\mathrm{im}(i)$

$q =$ the unique $(-1)^k$-quadratic form on $D = C_{k+1} \oplus C_k$ such that

$$q/C_{k\,1} = [\lambda_{k+1}] \quad , \quad q/C_k = [\lambda_k]$$

and such that IV.2(3) is satisfied.

Then E is a simple nondegenerate k-extension, since $\eta = \lambda_k$ on $D/\mathrm{im}(i) \cong C_k$; hence it represents an element

$$R^{new}((C,\lambda)) \text{ in } F_k^{new}((C,\lambda)) \quad .$$

Clearly the pair (F_k^{new},R^{new}) satisfies conditions (b) and (c) or (c') in II.3(iii) . Condition (a) is somewhat harder to verify.

---

*) and with the same restrictions; the proof will explain that .

Suppose that $f:(C,\lambda) \longrightarrow (C',\lambda')$ is a morphism in $\tilde{\mathcal{C}}_\lambda$ such that the underlying chain map $C \longrightarrow C'$ is a k-homotopy equivalence (see II.3(i)); then we must prove that $f_*: F_k^{new}((C,\lambda)) \longrightarrow F_k^{new}((C',\lambda'))$ is an isomorphism. A few 'observations' are now in order.

OBSERVATION 1: Let (C,λ) be in $\tilde{\mathcal{C}}_\lambda$, and let $\gamma \in Q(C)_0$ be a cycle homologous to λ (so that (C,γ) is another object in $\tilde{\mathcal{C}}_\lambda$); more specifically, suppose that an element $\sigma = (\sigma_n)_{n\in\mathbb{Z}}$ in $Q(C)_1 = \prod_{n\in\mathbb{Z}} \mathrm{Sel}(C_n)$ is given such that

$$\text{(i)} \qquad \delta_1(\sigma) = \gamma - \lambda \qquad ;$$

$$\text{(ii)} \qquad \sigma_n = 0 \qquad \text{for } n<k .$$

Then there is a 'sensible' group isomorphism

$$j: F_k^{new}((C,\lambda)) \longrightarrow F_k^{new}((C,\gamma)) .$$

(Proof: If $E = (D,i,f,b,q)$ is a simple nondegenerate k-extension of (C,λ), retain D,i,f,b and replace the $(-1)^k$-quadratic form q by

$$q+[f^*(\sigma_k)] \quad \in \quad \mathrm{coker}(\,1-(-1)^k T\,)$$

to get a simple nondegenerate k-extension of (C,γ) .)

Returning to the alleged isomorphism $f_*:F_k^{new}((C,\lambda)) \longrightarrow F_k^{new}((C',\lambda'))$, we find that Observation 1 does yield a candidate for an inverse homomorphism. Indeed, let $f^{-1}:C' \longrightarrow C$ be a chain map which is k-homotopy inverse to $f:C \longrightarrow C'$ (in loose notation). Choose a k-homotopy $(s_n: C'_{n-1} \longrightarrow C'_n)_{n\in\mathbb{Z}}$ from the identity to $f\cdot f^{-1}$ (so that $s_n = 0$ for $n \leq k$, and $\mathrm{id}+s_n\partial+\partial s_{n+1} = f_n\cdot f_n^{-1}$ for all n).

Let γ' be the cycle in $Q(C')_0$ obtained by pulling back $\lambda \in Q(C)_0$, using the chain map f^{-1} ; then f^{-1} can be regarded as a morphism

in $\tilde{\mathcal{E}}_\Lambda$, namely $f^{-1}:(C',\gamma') \longrightarrow (C,\lambda)$,
inducing $f_*^{-1}: F_k^{new}((C',\gamma')) \longrightarrow F_k^{new}((C,\lambda))$.
But Observation 1 above asserts the existence of a certain isomorphism

$$j: F_k^{new}((C',\lambda')) \longrightarrow F_k^{new}((C',\gamma'))$$

(notice that the k-homotopy $s = (s_n)$ yields a δ as in Observation 1 , namely $\delta = s\&\lambda'$; cf. proof of III.4).
Hence $f_*^{-1}\cdot j$ is a candidate for an inverse .

In order to prove that it is an actual inverse, we need to know more about the isomorphism j .

OBSERVATION 2: Let (C,λ) be in $\tilde{\mathcal{E}}_\Lambda$, $h: C \longrightarrow C$ a chain map , and $s = (s_n:C_{n-1} \longrightarrow C_n)_{n\in\mathbb{Z}}$ a k-homotopy from the identity to h ($s_n = 0$ for $n\leq k$). Put $\gamma = h^*(\lambda)$, so that h can be regarded as a morphism in $\tilde{\mathcal{E}}_\Lambda$,

$$h: (C,\gamma) \longrightarrow (C,\lambda) \quad ;$$

and let $j: F_k^{new}((C,\lambda)) \longrightarrow F_k^{new}((C,\gamma))$ be the isomorphism of Observation 1 , obtained using the k-homotopy s and the recipe from the proof of III.4 .
Then $h_*j: F_k^{new}((C,\lambda)) \longrightarrow F_k^{new}((C,\lambda))$ is the identity .

Proof: Fix a k-extension E of (C,λ) , but think of it as suggested in IV.3 , namely as an inclusion $(C,\lambda) \longrightarrow (C',\lambda')$ (with $C'_j = C_j$ for $j \neq k, k+1$ etc.). Choose a k-homotopy $\bar{s} = (\bar{s}_n: C'_{n-1} \longrightarrow C'_n)_n$ extending s , and with the property that $\bar{s}_{k+1}: C'_k \longrightarrow C'_{k+1}$ vanishes on the preferred complement of C_k in C'_k .
Thus $\bar{s}$ is a homotopy from the identity on C' to some chain map $\bar{h}$;

$\bar{h}$ extends h .

Put $\gamma' = \bar{h}^*(\lambda')$. Then the inclusion $(C,\gamma) \hookrightarrow (C',\gamma')$ can be considered as a k-extension of (C,γ) , say $E^?$. The diagram

$$\begin{tikzcd} (C,\gamma) \arrow[r, "h"] \arrow[d, hook] & (C,\lambda) \arrow[d, hook] \\ (C',\gamma') \arrow[r, "\bar{h}"] & (C',\lambda') \end{tikzcd}$$

shows that $h_*([E^?]) = [E]$ in $F_k^{new}((C,\lambda))$. On the other hand, the cycle $\gamma' \in Q(C')_0$ can also be expressed in terms of λ' and the homotopy $\bar{s}$, according to III.4 ; which proves that $j([E]) = E^?$. Hence $h_* j$ is the identity, q.e.d. .

Substituting $f \cdot f^{-1}$ for h in Observation 2 shows that $f_* \cdot f_*^{-1} \cdot j = id$; hence f_* is onto and f_*^{-1} is injective. By justice, f_*^{-1} is onto and f_* is injective, as required .

Condition II.3(iii)(a) is now supposed to have been verified. It remains to establish the universal property II.3(iii)(d) or (d') .

Let (F_k', R) be any other pair satisfying conditions II.3(iii)(a) - (d) ; fix an object (C,λ) in $\tilde{\mathcal{C}}_\Lambda$, and a k-extension $E = (D,i,f,b,q)$ of (C,λ) .

As in IV.3 , this defines an inclusion of (C,λ) in a bigger object (C',λ') (where $C_j' = C_j$ for $j \neq k, k+1$ and C_k' is split , $C_k' = C_k \oplus D/_{im(i)}$).

Let (C'',λ'') be the 'subobject' of (C',λ') defined by

$$C''_j = \begin{cases} C'_j = C_j & \text{if } j<k \\ D/\mathrm{im}(i) \subset C'_k & \text{if } j=k \\ 0 & \text{if } j>k \end{cases} .$$

Now (C'',λ'') is k-regular, so determines a characteristic element $R'((C'',\lambda'')) \in F'_k((C'',\lambda''))$; the inclusion $(C'',\lambda'') \longrightarrow (C',\lambda')$ maps this to an element in $F'_k((C',\lambda'))$. But $F'_k((C',\lambda')) \cong F'_k((C,\lambda))$ since the inclusion $(C,\lambda) \longrightarrow (C',\lambda')$ is a k-homotopy equivalence. The conclusion is that the k-extension E of (C,λ) determines an element $t(E)$ in $F'_k((C,\lambda))$.

With this in mind it is not hard to see that there is at most one transformation from F^{new}_k to F'_k mapping R^{new} to R' ; it must send the class of E in $F^{now}_k((C,\lambda))$ to $t(E) \in F'_k((C,\lambda))$.

To say that this is well determined is to say that $t(E) = 1$ whenever E is hyperbolic. This follows from condition II.3(iii)(c) or (c') for (F',R') and from

OBSERVATION 3: If E is a hyperbolic k-extension of (C,λ) , then there exists another s. nond. k-extension $\bar{E}$ of (C,λ) such that $\bar{E}\cdot E \cong \bar{E}\cdot E_0$, where E_0 is a 'very hyperbolic' k-extension of (C,λ) .

(Explanation and hint: $E_0 = (D,i,f,b,q)$ is 'very hyperbolic' if it is hyperbolic and $f=0$. Any s. nond. k-extension $\bar{E} = (\bar{D},\bar{i},\bar{f},\bar{b},\bar{q})$ with $\bar{f}$ surjective will do.)

The proof of proposition IV.12 is complete .

PROOF of IV.13:

To get from $G(P_{k-1})$ to $F_k^{new}((C(\tilde{B}),\lambda))$, suppose that $x \in G(P_{k-1})$ is represented by the pair (N_0^{2k}, j_0) (where $j_0 : P_{k-1}\times I \longrightarrow \partial N_0$ is a γ-embedding giving an open book decomposition for ∂N_0).

Substituting $P_{k-1}\times I$ for V in I.8(iii) , we can assume*) (performing surgeries on spheres in the interior of N_0 if necessary) that

$$N_0 = (P_{k-1}\times I)\times I \cup (\text{handles of index } k) \quad .$$

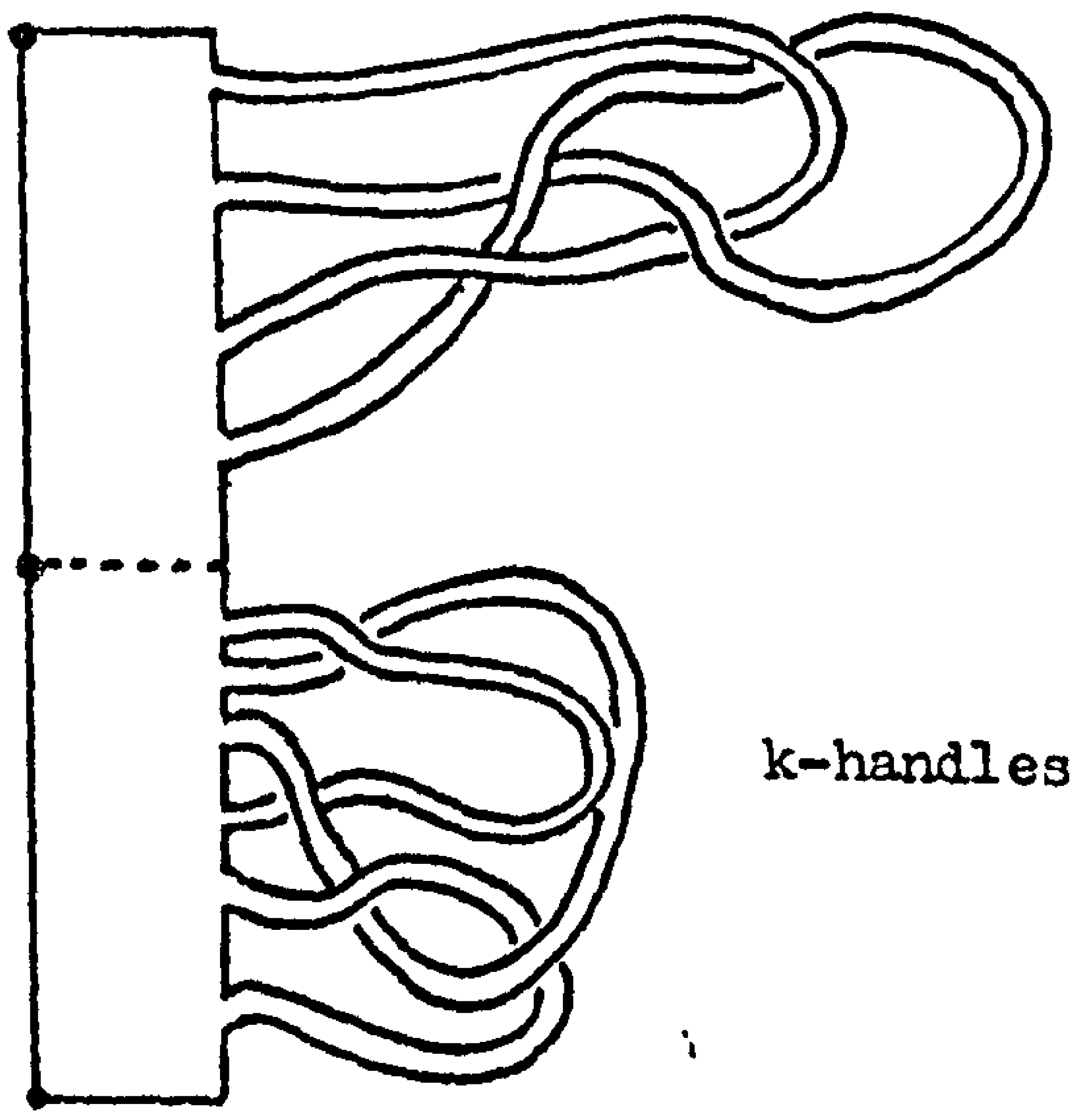
By construction of the γ-fattening $(P_n)_{n\ 0,1,..}$ we also have

$$P_k = (P_{k-1}\times I)\times I \cup (\text{handles of index } k)$$

(with different handles, of course).

A simple nondegenerate k-extension of $(C(\tilde{B}),\lambda)$ is now obtained by glueing P_k on top of N_0 and reading off homology groups .

Glueing takes place as in the picture:



k-handles of P_k

k-handles of N_0

and by 'reading off' is meant 'substituting homology groups for the terms in the diagram

$$\begin{tikzcd}
 & & C_k \arrow[dl] \arrow[dd, "i_k"] & C_{k+1} \arrow[l] \arrow[dd, "i_{k+1}"] & & \\
\cdots & C_{k-1} \arrow[l, dash, dashed] & & & C_{k+2} \arrow[ul] \arrow[dl] & \cdots \arrow[l, dashed] \\
 & & C'_k \arrow[ul, "{}'_k"] & C'_{k+1} \arrow[l, "\partial'_{k+1}"] & &
\end{tikzcd}$$

*) unless $k=2$; never mind.

from IV.3 . Here the top row is supposed to constitute the object $(C(\tilde{B}),\lambda)$, so that

$$C_j = H_j(P_j, P_{j-1}\times I)$$

and λ_j = sliding form (see III.1).

(Coefficients $\mathbb{Z}[\pi_1(B)]$ are understood.)

The rest of the diagram is

$$C'_k = H_k(P_k \cup N_0 \,,\, P_{k-1}\times(2\cdot I))$$

with the obvious splitting

$$C'_k = H_k(P_k, P_{k-1}\times I) \oplus H_k(N_0, P_{k-1}\times I)$$

so that i_k = inclusion of left summand ,

and λ'_k = sliding form again;

while C'_{k+1} and λ'_{k+1} and the maps i_{k+1} , ∂'_{k+1} are described (up to isomorphism, vaguely speaking) by the following procedure.

Notice that the homomorphism *)

$$\begin{array}{ccc} v:\ H_{k+1}(B,P_k) & \longrightarrow & H_{k+1}(B,P_k\cup N_0) \\ \wr\wr & & \wr\wr \\ \pi_{k+1}(B,P_k) & & \pi_{k+1}(B,P_k\cup N_0) \end{array}$$

is injective, and that coker(v) is f.g. free, with one generator for each k-handle in the decomposition of N_0 .

Let J be a complement of im(v) in $H_{k+1}(B,P_k\cup N_0)$.

Put

$$C'_{k+1} = C_{k+1} \oplus J$$

$$i_{k+1} = \text{inclusion of left summand}$$

*) If L is any γ-manifold, write $H_n(B,L)$ etc. for $H_n(\gamma)$, where $\gamma: L \longrightarrow B$ is the classifying map for the normal bundle of L .

and ∂'_{k+1} = composite

$$\begin{tikzcd} & C'_{k+1} = C_{k+1} \oplus J = H_{k+1}(P_{k+1}, P_k \times I) \oplus J \arrow[d, "\varphi \oplus \text{inclusion}"] \\ C'_k = H_k(P_k \cup N_0, P_{k-1} \times (2 \cdot I)) & H_{k+1}(B, P_k \cup N_0) \arrow[l, "\text{boundary}"'] \end{tikzcd}$$

(where φ in turn is the composite of the homomorphism $H_{k+1}(P_{k+1}, P_k \times I) \longrightarrow H_{k+1}(B, P_k \times I) \cong H_{k+1}(B, P_k)$ induced by the classifying map $P_{k+1} \longrightarrow B$, with the homomorphism $H_{k+1}(B, P_k) \longrightarrow H_{k+1}(B, P_k \cup N_0)$).

To get $[\lambda'_{k+1}]$ (recall that only a coset in $\operatorname{coker}(1 - (-1)^k T)$ is required, not a sesquilinear form λ'_{k+1} on C'_{k+1}), notice that the term $H_{k+1}(B, P_k \cup N_0) \cong \pi_{k+1}(B, P_k \cup N_0)$ appearing in the definition of ∂'_{k+1} is equipped with a $(-1)^k$-quadratic form (consisting of mutual and self-intersection numbers). Pulling it back to C'_{k+1} (using the homomorphism $\varphi \oplus$inclusion) gives $[\lambda'_{k+1}]$, by definition .

This is a complete k-extension (and its isomorphism class does not depend on the choice of a 'complement' J).

To recapitulate: We started with $x \in G(P_{k-1})$, chose a fixed representative (N_0, j_0) with certain agreeable properties, and obtained an isomorphism class of simple nondegenerate k-extensions, of $(C(\tilde{B}), \lambda)$, hence an element in $F_k^{new}((C(\tilde{B}), \lambda))$.
(The fact that $j_0 : P_{k-1} \times I \longrightarrow \partial N_0$ gives an open book decomposition for ∂N_0 was used twice: first to get the representative (N_0, j_0)

with agreeable properties, and then to show that the resulting k-extension is simple nondegenerate.)

Suppose that (N_1, j_1) is another representative for the same $x \in G(P_{k-1})$, with the same agreeable properties (so that

$$N_1 \cong (P_{k-1} \times I) \times I \cup (\text{k-handles}) ,$$

$$j_1 : P_{k-1} \times I \cong (P_{k-1} \times I) \times \{0\} \hookrightarrow (P_{k-1} \times I) \times I \cup (\text{k-handles})).$$

Does the pair (N_1, j_1) determine the same element in $F_k^{new}((C(\widetilde{B}), \lambda))$ as (N_0, j_0) ?

It does, by the following argument. Let Z^{2k+1} be a bordism modulo boundary (a (B,f)-bordism, or γ-bordism) from N_0 to N_1 as in I.6(ii) (here again, $G(P_{k-1})$ is treated as a subset of $\mathcal{S}(P_{k-1} \times I)$; Z exists because (N_0, j_0) and (N_1, j_1) represent the same element in $G(P_{k-1})$). Surgery can be performed on spheres in the interior of Z until it has the usual agreeable properties . (Then the inclusions $N_0 \hookrightarrow Z$, $N_1 \hookrightarrow Z$ are (k-1)-equivalences.)

By Morse theory or otherwise ([WALL 3, IV , 5.1]) , a Morse function 'modulo boundary' $m: Z \longrightarrow [0,1] = I$ can be found (with $m(N_0) = 0$, $m(N_1) = 1$) such that all critical points have index k or $k+1$.

The Morse function can be so rearranged ([MILNOR 1]) that all critical points of index k have values $< \frac{1}{2}$, say, and those of index $k+1$ have values $> \frac{1}{2}$; in which case we put $N_{\frac{1}{2}} = m^{-1}(\frac{1}{2})$.

Conclusion: There exists a third representative $(N_{\frac{1}{2}}, j_{\frac{1}{2}})$ for $x \in G(P_{k-1})$ such that

(i) $(N_{\frac{1}{2}}, j_{\frac{1}{2}})$ is obtained from (N_0, j_0) by performing surgery on elements of $\pi_k(B, N_0)$ ($= \pi_k(\gamma)$, where γ goes from N_0 to B ; notation as in [WALL 1 , Thm.1.1] . Forgetting about B , one might say that surgery is performed on framed $(k-1)$-spheres in the interior of N_0) ;

(ii) $(N_{\frac{1}{2}}, j_{\frac{1}{2}})$ can also be obtained from (N_1, j_1) through surgery on elements of $\pi_k(B, N_1)$.

Hence only the following question remains:

How does a single surgery on an element y in $\pi_k(B, N_0)$ (as in (i)) affect the associated simple nondegenerate k-extension of $(C(\widetilde{B}), \lambda)$?

This boils down to:

How does such a single surgery affect the decomposition of N_0 as

$$N_0 \cong (P_{k-1} \times I) \times I \cup (\text{k-handles}) \qquad ?$$

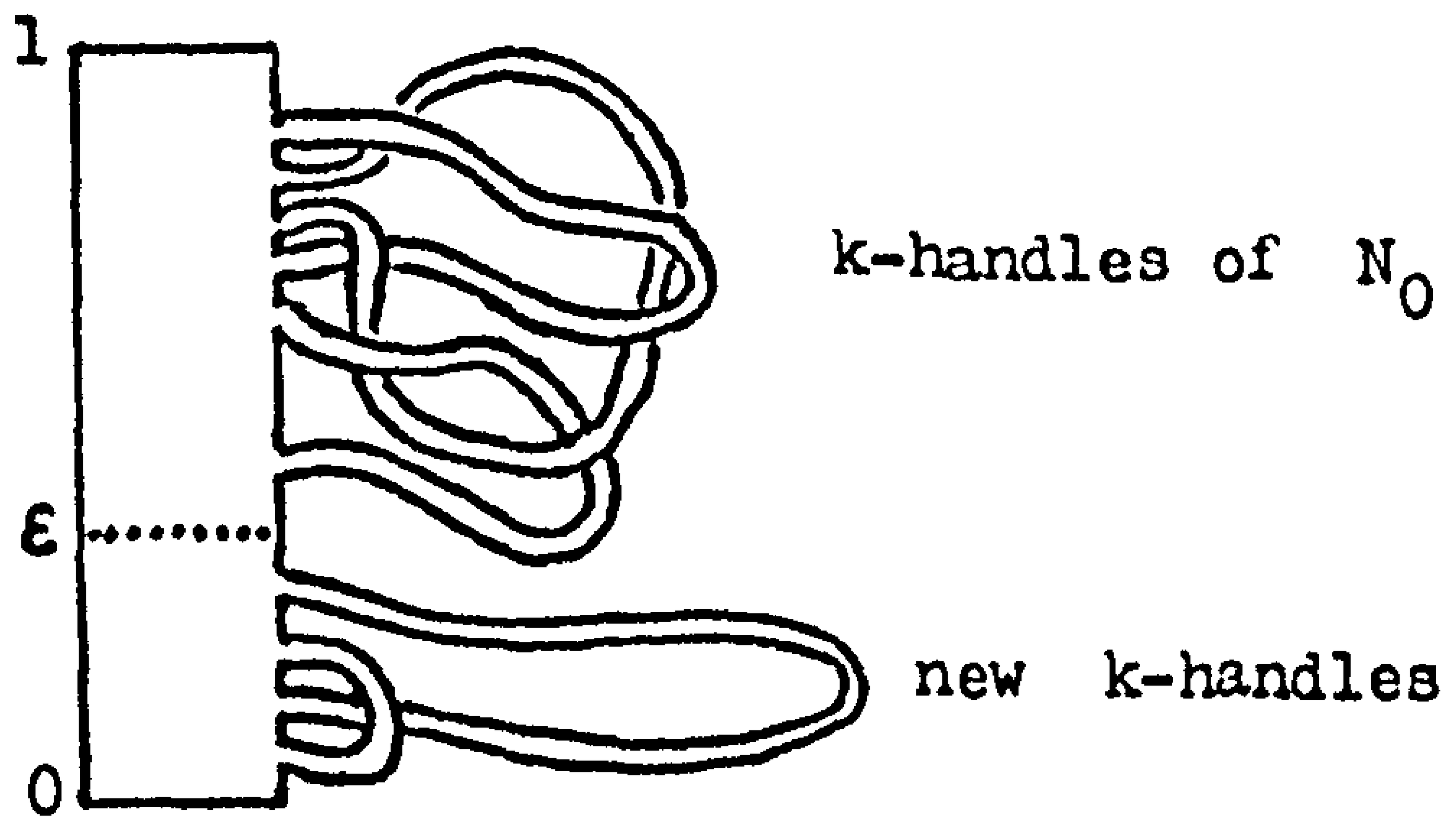
To answer this, bear in mind that, for any small real number $\varepsilon > 0$,

$$\pi_k(B, P_{k-1} \times [0,\varepsilon]) \cong \pi_k(B, P_{k-1} \times I) \xrightarrow[j_{0*}]{} \pi_k(B, N_0)$$

is onto . Choose $\bar{y} \in \pi_k(B, P_{k-1} \times [0,\varepsilon])$ mapping to $y \in \pi_k(B, N_0)$.

One finds that the effect of the surgery is to add two further k-handles to the picture: one representing the class $\bar{y} \in \pi_k(B, P_{k-1} \times [0,\varepsilon])$, the other representing the zero class, and the two linked .

Picture:



If ε has been chosen sufficiently small, the two new k-handles will not interfere with the original ones.

'Reading off' now yields the desired result: namely, if E is the simple nondegenerate k-extension associated with (N_0, j_0), then the simple nondegenerate k-extension obtained after surgery has the form $E \cdot E_0$, where E_0 is hyperbolic (of 'rank 2' in this case).

Unfortunately, this does not prove IV.13 ; we have only just constructed a map $\alpha : G(P_{k-1}) \longrightarrow F_k^{new}((C(\tilde{B}), \lambda))$ (which is clearly a group homomorphism).

It is , however, fairly easy to read the proof backwards; this works when $k > 2$ and gives a homomorphism from $F_k^{new}((C(\tilde{B}), \lambda))$ – not 'a priori' to $G(P_{k-1})$, but – to the monoid $\mathcal{S}(P_{k-1} \times I)$ (see I.6 and I.8(ii)), which at least contains $G(P_{k-1})$.
Call this homomorphism β ; then $\beta \cdot \alpha$ equals the inclusion $G(P_{k-1}) \hookrightarrow \mathcal{S}(P_{k-1} \times I)$, which proves that α is injective if $k > 2$.
If $k > 3$, the s-cobordism theorem can be exploited (together with the fact that the k-extensions under consideration are simple nondegenerate); it shows that α is onto, which completes the proof.

V. MORE PROOFS

CONVENTIONS: The tensor product (over $\mathbb{Z}$, say) of two chain complexes C, D is defined by $(C\otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j$,

$$\partial^{C\otimes D}: (C\otimes D)_n \longrightarrow (C\otimes D)_{n-1}$$

$$\partial^{C\otimes D}/_{C_i\otimes D_j} = \partial^C \otimes id + (-1)^i id\otimes \partial^D .$$

I is the cellular chain complex of the standard 1-simplex, or unit interval, with two 0-cells; viz., $\{0\}$ (the 1^{st} face) and $\{1\}$ (the 0^{th} face). Usually complexes of the form $C\otimes I$ (rather than $I\otimes C$) are considered; hence the definition of ΣC , the suspension of C , involves <u>no</u> sign changes. Somewhat inconsistent with that is the notion of chain homotopy as used in III.4 .

V.I PROOF of the 'proof' of III.4 :

Assume first that the homotopy $s = (s_n)$ is concentrated in one dimension, $s_n = 0$ for $n \neq j+1$. Then

$$f_j + \partial s_{j+1} = g_j$$

$$f_{j+1} + s_{j+1}\partial = g_{j+1} ,$$

and $f_n = g_n$ for $n \neq j$, $j+1$.

Hence $[g_j(\lambda_j^!) - f_j(\lambda_j^!)]\,(x,y)$

$$= \lambda_j^!((f_j + \partial s_{j+1})(x) , (f_j + \partial s_{j+1})(y)) - \lambda_j^!(f_j(x),f_j(y)) ,$$

which can be further expanded; a similar expansion exists for $[g_{j+1}(\lambda_{j+1}^!) - f_{j+1}(\lambda_{j+1}^!)]\,(x,y)$, and one finds without too much difficulty that

$$f^*(\lambda') + \delta_1(s\&\lambda') = g^*(\lambda') ,$$

provided $s\&\lambda'$ is defined as in III.4 .
Now for the general case: An arbitrary chain homotopy s from f to g can be broken up into a succession of chain homotopies each of which is concentrated in one dimension (possibly an infinite succession); iteration of the construction for the special case above then shows that the formula in III.4 is correct . (Observe, however, that $s\&\lambda'$ depends not only on λ' and the sequence $(s_n : C_{n-1} \longrightarrow C_n)$, but also on f and/or g . This slightly complicates the verification.)

Having dealt with $H_0(Q(-))$, we can dispose of $H_1(Q(-))$ in the following manner.

V.2 OBSERVATION: There is a natural isomorphism of chain complexes

$$Z\colon Q(\Sigma C) \longrightarrow \Sigma(Q(C)) \qquad (\ \Sigma \text{ denotes suspension}).$$

Namely, if $\lambda = (\lambda_n)_{n\in\mathbb{Z}} \in Q(\Sigma C)_p$,

$$\lambda_n\colon (\Sigma C)_n \times (\Sigma C)_n = C_{n-1} \times C_{n-1} \longrightarrow \Lambda ,$$

we put $Z(\lambda) = \gamma \in \Sigma(Q(C))_p = Q(C)_{p-1}$
with $\gamma_n = (-1)^{n+p} \cdot \lambda_{n+1} : C_n \times C_n \longrightarrow \Lambda$.

It seems plausible that a chain homotopy from $f^*\colon Q(C') \longrightarrow Q(C)$ to $g^*\colon Q(C') \longrightarrow Q(C)$ can be constructed from these scraps, but nothing of this sort will be needed. But in a different respect, more precision is desirable (for the proof of III.9).
Think of λ' as a point in the simplicial abelian group $K(Q(C'))$. The two (homotopic) maps $f,g\colon C \longrightarrow C'$ induce homomorphisms of simplicial abelian groups $f^*, g^*\colon K(Q(C')) \longrightarrow K(Q(C))$,

and III.4 states that $f^*(\lambda')$ and $g^*(\lambda')$ belong to the same component of $K(Q(C))$; for $s\&\lambda'$ is a path connecting the two. Fears may now arise that it is not the best possible path. To savour these, regard the chain homotopy s as a chain map from $C\otimes I$ to C' , restricting to f and g on $C\otimes\{0\}$ and $C\otimes\{1\}$ respectively.
Then $s:C\otimes I \longrightarrow C'$ induces $s^*: K(Q(C')) \longrightarrow K(Q(C\otimes I))$.
Further, the inclusions $C\otimes\{0\} \longrightarrow C\otimes I$, $C\otimes\{1\} \longrightarrow C\otimes I$ induce maps $K(Q(C\otimes I)) \longrightarrow K(Q(C\otimes\{0\}))$, $K(Q(C\otimes I)) \longrightarrow K(Q(C\otimes\{1\}))$, sending the point $s^*(\lambda') \in K(Q(C\otimes I))$ to $f^*(\lambda')$ and $g^*(\lambda')$ respectively .
Let X_I be <u>the</u> universal covering space of the component of $K(Q(C\otimes I))$ containing $s^*(\lambda')$, with reference to the base point $s^*(\lambda')$.
Similarly, X_0 and X_1 denote the universal covering spaces associated with $f^*(\lambda') \in K(Q(C\otimes\{0\}))$ and $g^*(\lambda') \in K(Q(C\otimes\{1\}))$ respectively. X_I , X_0 and X_1 are themselves based, and the group of covering transformations is the same for all three (by III.4).
The diagram

$$\begin{tikzcd} & K(Q(C\otimes\{0\})) \\ K(Q(C\otimes I)) \arrow[ur] \arrow[dr] & \\ & K(Q(C\otimes\{1\})) \end{tikzcd}$$

has a preferred lifting to a diagram

$$\begin{tikzcd} & X_0 \\ X_I \arrow[ur, "i_0^*"'] \arrow[dr, "i_1^*"'] & \\ & X_1 \end{tikzcd}$$

(in which both maps are base-point preserving).Furthermore, the map (not base-point preserving) $K(Q(C\otimes\{0\})) \longrightarrow K(Q(C\otimes I))$ induced by the projection $C\otimes I \longrightarrow C\otimes\{0\}$ also has a preferred lifting

$$pr^*: X_0 \longrightarrow X_I \quad ,$$

namely the one such that $i_0^* \cdot pr^* = id$.

The map $i_1^* \cdot pr^*: X_0 \longrightarrow X_1$ is a lifting of the obvious identification $K(Q(C\otimes\{0\})) \cong K(Q(C\otimes\{1\}))$; hence it maps the base point in X_0 to a point z in X_1 lying above $f^*(\lambda') \in K(Q(C\otimes\{1\})) \cong K(Q(C\otimes\{0\}))$. Giving such a point amounts to specifying a preferred homotopy class of paths connecting $f^*(\lambda')$ with the base-point $g^*(\lambda')$ of $K(Q(C\otimes\{1\}))$. There we are: this is the 'best possible' path class .

V.3 PROPOSITION: The path $s\&\lambda'$ represents the best possible path class .

Proof: The concatenation of $s\&\lambda'$ with the inverse of a 'best possible' path gives a closed loop at $f^*(\lambda')$; i.e. an element $Lp(s)$ in $\pi_1(K(Q(C));f^*(\lambda'))$, which must be proved equal to 1 . A nulhomotopy of $Lp(s)$ can be obtained simply by observing that the homotopy s is itself 'nulhomotopic' .

In detail, let $\bar{f}: C\otimes I \longrightarrow C'$ be any chain map restricting to f on $C\otimes\{0\}$; and let $\bar{s}$ be any homotopy from $\bar{f}$ to $\bar{g}$ (say) restricting to s on $C\otimes\{0\}$, and restricting to a constant homotopy 'const.' on $C\otimes\{1\}$. As usual, the inclusions $C\otimes\{0\} \longrightarrow C\otimes I$, $C\otimes\{1\} \longrightarrow C\otimes I$ induce homomorphisms

$$\begin{tikzcd}[row sep=small] & \pi_1(K(Q(C\otimes\{0\}))) \\ \pi_1(K(Q(C\otimes I))) \arrow[ur] \arrow[dr] & \\ & \pi_1(K(Q(C\otimes\{1\}))) \end{tikzcd}$$

(base-points may be suppressed since all groups are abelian) .

By naturality of the construction in III.4 , the upper homomorphism maps $Lp(\bar{s})$ to $Lp(s)$, and the lower maps $Lp(\bar{s})$ to Lp(const.) . But the two homomorphisms are isomorphisms by III.4 , and clearly Lp(const.) = 1 . Hence $Lp(s) = 1$, q.e.d. .

V.4 PROOF of III.9 .

In the notation of III.9 and the paragraph preceding it, let $\mathcal{D}_0$ be the image of $\tilde{\mathcal{C}}_\Lambda$ in $\mathcal{D}$; thus the inclusion $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ is an equivalence of categories. The triangle

$$\begin{tikzcd} & \mathcal{H}\tilde{\mathcal{C}}_\Lambda \arrow[d, "J"] \\ \tilde{\mathcal{C}}_\Lambda \arrow[ur] \arrow[r] & \mathcal{D} \end{tikzcd}$$

can be replaced by

$$\begin{tikzcd} & \mathcal{H}\tilde{\mathcal{C}}_\Lambda \arrow[d, "J"] \\ \tilde{\mathcal{C}}_\Lambda \arrow[ur] \arrow[r] & \mathcal{D}_0 \end{tikzcd} ,$$

in which all three categories have the same class of objects.

A functor $V: \mathcal{D}_0 \longrightarrow \mathcal{H}\tilde{\mathcal{C}}_\Lambda$ inverse to $J: \mathcal{H}\tilde{\mathcal{C}}_\Lambda \longrightarrow \mathcal{D}_0$ is now asked for.

Such a functor arises rather naturally from a 'realization procedure' for the simplicial spaces $K(Q(C))$. I shall explain this realization procedure in Part 1 below; Part 2 will show how the required funtor V can be obtained from it, and Part 3 will show that $V \cdot J$ and $J \cdot V$ are the identity functors on $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$ and $\mathcal{D}_0$ respectively .

PART 1 (The realization procedure):

Fix a chain complex C in $\mathcal{C}_\Lambda$. Let Δ_n denote the cellular chain complex of the standard n-simplex (which is a chain complex of abelian groups); in particular, $\Delta_1 = I$.

Let $Y(C)$ be the simplicial abelian group such that e.g.

$$Y(C)_n \; := \; \text{abelian group of cycles in} \quad Q(C \otimes \Delta_n)_0 \quad .$$

(The tensor product $C \otimes \Delta_n$ is to be taken over $\mathbb{Z}$, with the conventions laid down at the beginning of this chapter; $C \otimes \Delta_n$ is then a chain complex in $\mathcal{C}_\Lambda$ for each n .Each order-preserving map $\{0,1,\ldots,m\} \longrightarrow \{0,1,\ldots,n\}$ gives a cellular map from the standard m-simplex to the standard n-simplex , hence a chain map from Δ_m to Δ_n ; tensoring with the identity on C , a chain map $C \otimes \Delta_m \longrightarrow C \otimes \Delta_n$. Since Q is a contravariant functor, we obtain the required simplicial operator $Y(C)_n \longrightarrow Y(C)_m$ associated with the order-preserving map .)

Note that

$$\begin{aligned} \text{0-skeleton of } Y(C) &= \text{0-skeleton of } K(Q(C)) \\ &= \text{group of cycles in } Q(C)_0 \quad . \end{aligned}$$

We shall construct (in three steps) a homomorphism of simplicial abelian groups

$$R\colon K(Q(C)) \longrightarrow Y(C) \quad ,$$

restricting to the identification above on the 0-skeletons.

Step 1 . For $p \geqslant 1$, let D_p be the chain complex (of abelian groups)

$$(D_p)_n = \begin{cases} \mathbb{Z} & \text{if} \quad n = p \quad \text{or} \quad n = p-1 \\ 0 & \text{otherwise} \; ; \end{cases}$$

$$id_{\mathbb{Z}} = \partial_p\colon (D_p)_p \longrightarrow (D_p)_{p-1} \qquad .$$

Thus $C \otimes_{\mathbb{Z}} D_p$ is the $(p-1)$-fold suspension of the cone on C .

Given a chain (not necessarily a cycle) $\phi = (\phi_n)_{n\in\mathbb{Z}}$ in $Q(C)_p = \prod_{n\in\mathbb{Z}} \mathrm{Sel}(C_n)$ (see II.1) , we shall associate with it a cycle $r(\phi) = (r(\phi)_n)_{n\in\mathbb{Z}}$ in

$$Q(C \otimes D_p)_0 = \prod_{n\in\mathbb{Z}} \mathrm{Sel}(C \otimes D_p)_n$$

as follows.

Using the differential δ in $Q(C)$, write $\lambda = \delta_p(\phi)$ and define the sesquilinear form $r(\phi)_n$ on $(C \otimes D_p)_n \cong C_{n-p} \oplus C_{n-(p-1)}$ by

$$r(\phi)_n(x,y) = \begin{cases} (-1)^{(n-1)p} \cdot T^p \phi_{n-p}(x,y) & \text{for } x,y \in C_{n-p} \\ (-1)^{(n-1)(p-1)} \cdot T^{(p-1)} \lambda_{n-(p-1)}(x,y) & \text{for } x,y \in C_{n-(p-1)} \\ (-1)^{(n-1)} \lambda_{n-p}(x, \partial y) & \text{for } x \in C_{n-p} ,\ y \in C_{n-(p-1)} \\ 0 & \text{for } x \in C_{n-(p-1)} ,\ y \in C_{n-p} . \end{cases}$$

(Here T denotes the transposition involution ; see p.II.1.)
Verifying that $r(\phi) = (r(\phi)_n)_{n\in\mathbb{Z}}$ is a cycle is a rather tedious task, and will be omitted.

Step 2: We construct a chain map

$$\hat{R} : N(K(Q(C))) \longrightarrow N(Y(C))$$

where N is the Dold-Kan functor from simplicial abelian groups to chain complexes (see p.III.4).

As a preparation, let $f_p\colon \Delta_p \longrightarrow D_p$ be the collapsing map. (The abelian group $(\Delta_p)_n$ is freely generated by the set of subsets of $\{0,1,\ldots,p\}$ having cardinality $n+1$; the collapsing map is obtained by mapping the generators

$$\{0,1,\ldots,p\} \in (\Delta_p)_p \quad \text{to} \quad 1 \in (D_p)_p = \mathbb{Z} ,$$
$$\{1,2,\ldots,p\} \in (\Delta_p)_{p-1} \quad \text{to} \quad 1 \in (D_p)_{p-1} = \mathbb{Z} ,$$

and all other standard generators to 0 .)

Then f_p induces maps $\quad id \otimes f_p : C \otimes \Delta_p \longrightarrow C \otimes D_p$

and $\quad (id \otimes f_p)^*$: group of cycles in $Q(C \otimes D_p)_0$

$\longrightarrow$ group of cycles in $Q(C \otimes \Delta_p)_0 = Y(C)_p$.

Note that $N(K(Q(C))) \cong Q(C)^+$, in the notation of p.III.5 . Hence we may define

$$\hat{R}\colon N(K(Q(C)))_0 \longrightarrow N(Y(C))_0$$

to be the identification

$$N(K(Q(C)))_0 \cong \text{group of cycles in } Q(C)_0 \cong N(Y(C))_0 ,$$

and
$$\hat{R}\colon \underset{\substack{\wr\wr \\ Q(C)_p}}{N(K(Q(C)))_p} \longrightarrow N(Y(C))_p \qquad (\text{for } p \geq 1)$$

by
$$\sigma \longmapsto (id \otimes f_p)^*(r(\sigma)) \in N(Y(C))_p \subset Y(C)_p .$$

$\hat{R}$ is easily seen to be a chain map.

<u>Step 3</u>: The Dold-Kan equivalence of categories transforms the chain map $\hat{R}$ from step 2 into a homomorphism of simplicial abelian groups

$$R\colon K(Q(C)) \longrightarrow Y(C) .$$

PART 2: Construction of the functor $V\colon \mathcal{D}_0 \longrightarrow \mathcal{H}\tilde{\mathcal{E}}_\Lambda$.

Let $f\colon(C,\lambda) \longrightarrow (C',\lambda')$ be a morphism in $\mathcal{D}_0$. Such a morphism f is really a homotopy class of pairs, each pair consisting of a chain map $f_? : C \longrightarrow C'$ and a preferred homotopy class of paths in $K(Q(C))$ from λ to $f_?^*(\lambda')$ (see the definition of $\mathcal{D}$ on p.III.8).

Now the realization map R allows us to replace the path class in $K(Q(C))$ from λ to $f_?^*(\lambda')$ by a path class in $Y(C)$ from λ to $f_?^*(\lambda')$. Since $Y(C)$ is a simplicial abelian group, it has the Kan extension property; so the path class can be represented by a 1-simplex in $Y(C)$. By definition, this is a cycle $\curlyvee$ in $Q(C\otimes I)$, restricting to λ and $f_?^*(\lambda')$ in $Q(C\otimes\{0\})$ and $Q(C\otimes\{1\})$ respectively .

So we can define $V(f)$ (a morphism in $\mathcal{H}\tilde{\mathcal{E}}_\Lambda$ from (C,λ) to (C,λ')) to be the composite

$$(C,\lambda) \cong (C\otimes\{0\},\lambda) \xrightarrow[i_0]{} (C\otimes I,\curlyvee) \xrightarrow[i_1^{-1}]{} (C,f_?^*(\lambda')) \xrightarrow[j]{} (C',\lambda') .$$

(Here i_0 , i_1 denote the obvious inclusions; they are invertible in $\mathcal{H}\tilde{\mathcal{E}}_\Lambda$. The morphism j on the right comes from a morphism in $\tilde{\mathcal{E}}_\Lambda$ which agrees with $f_?$ on the chain complex level .)

$V(f)$ is easily seen to be well defined.

PART 3:

$V\cdot J$ is the identity on $\mathcal{H}\tilde{\mathcal{E}}_\Lambda$ because the diagram

$$\begin{tikzcd} & \mathcal{H}\tilde{\mathcal{E}}_\Lambda \arrow[d, "V\cdot J"] \\ \tilde{\mathcal{E}}_\Lambda \arrow[ur] \arrow[r] & \mathcal{H}\tilde{\mathcal{E}}_\Lambda \end{tikzcd}$$

is commutative (so the universal property of $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$ applies).

To see that $J \cdot V$ is the identity on $\mathcal{D}_0$, let $f:(C,\lambda) \longrightarrow (C',\lambda')$ be a morphism in $\mathcal{D}_0$. Again, this may be represented by a pair, consisting of a chain map $f_?:C \longrightarrow C'$ and a 1-simplex w in $K(Q(C))$, connecting λ with $f_?^*(\lambda')$; or in other words, a 'homology' w from λ to $f_?^*(\lambda')$. The functor V essentially replaces the homology w by a structure on $C \otimes I = C \otimes \Delta_1$, namely a cycle $R(w)$ in $Q(C \otimes I)_0$ (the explicit formula for $R(w)$ can be recovered from PART 1 of this proof).

Some pondering shows that the functor J replaces the structure $R(w)$ by a path (or path class) in $K(Q(C))$ again, with the same endpoints λ and $f_?(\lambda')$; this is what was termed the 'best possible' path class (in relation to the cycle $R(w)$) in the discussion preceding V.3 . Hence, by V.3 , we have an explicit description for the new path class in $K(Q(C))$, joining λ and $f_?^*(\lambda')$: it is represented by $s\&R(w)$, in the notation of Lemma III.4 (provided we let s be the standard homotopy from $i_0:C \cong C \otimes \{0\} \longrightarrow C \otimes I$ to $i_1:C \longrightarrow C \otimes I$). So it only remains to be seen that the 1-simplices w and $s\&R(w)$ are homotopic modulo endpoints. This computation is left to the reader.

The following proposition, which ought to have been stated earlier, may simplify it a little:

V.5 PROPOSITION: For any chain complex C in $\tilde{\mathcal{C}}_\Lambda$, the homology groups of $Q(C)$ are Z_2-vector spaces .

Proof: The homomorphisms

$$\begin{array}{rcccc} & \pi_{\mathrm{Sel}(C_n)} & & \pi_{\mathrm{Sel}(C_n)} \\ & \parallel & & \parallel \\ s_p : & Q(C)_{p-1} & \longrightarrow & Q(C)_p \\ & \lambda & \longmapsto & (-1)^{p-1}\lambda \end{array}$$

constitute a chain homotopy from 0 to $2\cdot\mathrm{id}$.

The proof of III.9 is complete, and we may turn to proving II.4 and II.7 . The central idea is in

V.6 LEMMA. let C be a chain complex in $\mathcal{C}_\Lambda$, and λ a cycle in $Q(C\otimes I)_0$, λ^0 its restriction to $C\otimes\{0\}$ (i.e. $\lambda^0 = i_0^*(\lambda)$, where $i_0^* : Q(C\otimes I) \longrightarrow Q(C\otimes\{0\})$ is induced by the inclusion i_0). Put $\lambda^0\otimes I = pr^*(\lambda^0)$, where $pr: C\otimes I \longrightarrow C \cong C\otimes\{0\}$ is the projection (just as in II.7) . Then there is a natural bijection (usually not a monoid homomorphism)

$$j: \mathcal{E}_k((C\otimes I,\lambda)) \longrightarrow \mathcal{E}_k((C\otimes I, \lambda^0\otimes I))$$

(cf. IV.10) so that the diagram

$$\begin{tikzcd}
\mathcal{E}_k((C\otimes I,\lambda)) \arrow[rr, "j"'] & & \mathcal{E}_k((C\otimes I,\ \lambda^0\otimes I)) \\
& \mathcal{E}_k((C\otimes\{0\},\lambda^0)) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

is commutative.

('Natural' refers to the following. If $f:D \longrightarrow C$ is a chain map, then the diagram

$$\begin{tikzcd}
\mathcal{E}_k(D\otimes I,(f\otimes I)^*(\lambda)) \arrow[r] \arrow[d] & \mathcal{E}_k((D\otimes I,f^*(\lambda^0)\otimes I)) \arrow[d] \\
\mathcal{E}_k((C\otimes I,\lambda)) \arrow[r] & \mathcal{E}_k((C\otimes I,\lambda^0\otimes I))
\end{tikzcd}$$

is commutative .)

Proof: Let $s = (s_n)$ be a (or the) deformation retraction deforming the chain complex I into its subcomplex $\{0\}$. Upon tensoring with C, this gives a deformation retraction s^C deforming $C\otimes I$ into $C\otimes\{0\}$. Write σ for $s^C\&\lambda$ (cf. proof of III.4); then

$$\lambda + \delta_1(\sigma) \quad = \quad \lambda^0\otimes I \qquad\qquad \text{in } Q(C\otimes I)_0 \quad .$$

In this situation, the recipe from the proof of IV.12 (Observation 1) can be used. That is,

$$j:\ \mathcal{E}_k((C\otimes I,\lambda)) \longrightarrow \mathcal{E}_k((C\otimes I,\ \lambda^0\otimes I))$$

takes a k-extension $E = (D,i,f,b,q)$ of $(C\otimes I, \lambda)$, not necessarily simple nondegenerate, and replaces q by

$$q + [f^*(\delta_k)] \quad \in \quad \mathrm{coker}(\,1 - (-1)^k T\,) \ .$$

In the present case, due to weaker assumptions, this construction preserves neither products nor units (e.g., simple nondegenerate extensions) ; but it leaves hyperbolic k-extensions hyperbolic, which is essentially the reason why it leads to a well-defined bijection j. Since the homotopy s^C vanishes on $C\otimes\{0\}$, so does $s^C \& \lambda$, which establishes the commutative triangle; naturality is also clear.

V.7 COROLLARY: Let λ be a cycle in $Q(C\otimes I)_0$, and $\lambda^0 \in Q(C\otimes\{0\})_0$ its restriction to $C\otimes\{0\}$. Then the homomorphism

$$F_k((C\otimes\{0\},\lambda^0)) \longrightarrow F_k((C\otimes I,\lambda))$$

(induced by the inclusion) is injective. In fact,

$$\mathcal{E}_k((C\otimes\{0\},\lambda^0)) \longrightarrow \mathcal{E}_k((C\otimes I,\lambda))$$

is injective, and has a natural left inverse (not a monoid homomorphism, just a set map).

Proof: The natural left inverse is the composite

$$\mathcal{E}_k((C\otimes I,\lambda)) \longrightarrow \mathcal{E}_k((C\otimes I,\lambda^0\otimes I)) \xrightarrow[\text{projection}]{} \mathcal{E}_k((C\otimes\{0\},\lambda^0)) \ .$$

V.8 COROLLARY: Let λ, γ be two cycles in $Q((C\otimes I))_0$;
put $\lambda^0 := i_0^*(\lambda)$, $\lambda^1 := i_1^*(\lambda)$, $\gamma^0 := i_0^*(\gamma)$,
$\gamma^1 := i_1^*(\gamma)$ (where $i_0\colon C\otimes\{0\}\cong C \hookrightarrow C\otimes I$, $i_1\colon C\otimes\{1\}\cong C \hookrightarrow C\otimes I$ are inclusions).

By V.7 , $F_k((C,\lambda^0))$ and $F_k((C,\lambda^1))$ may be regarded as subgroups of $F_k((C\otimes I,\lambda))$; similarly, $F_k((C,\gamma^0))$ and $F_k((C,\gamma^1))$ as subgroups of $F_k((C\otimes I,\gamma))$.

Now assume that $\lambda^0 = \gamma^0$ in $Q(C)_0$.

Let $x \in F_k((C,\lambda^0)) = F_k((C,\gamma^0))$.

The following statements (i) , (ii) are then equivalent:

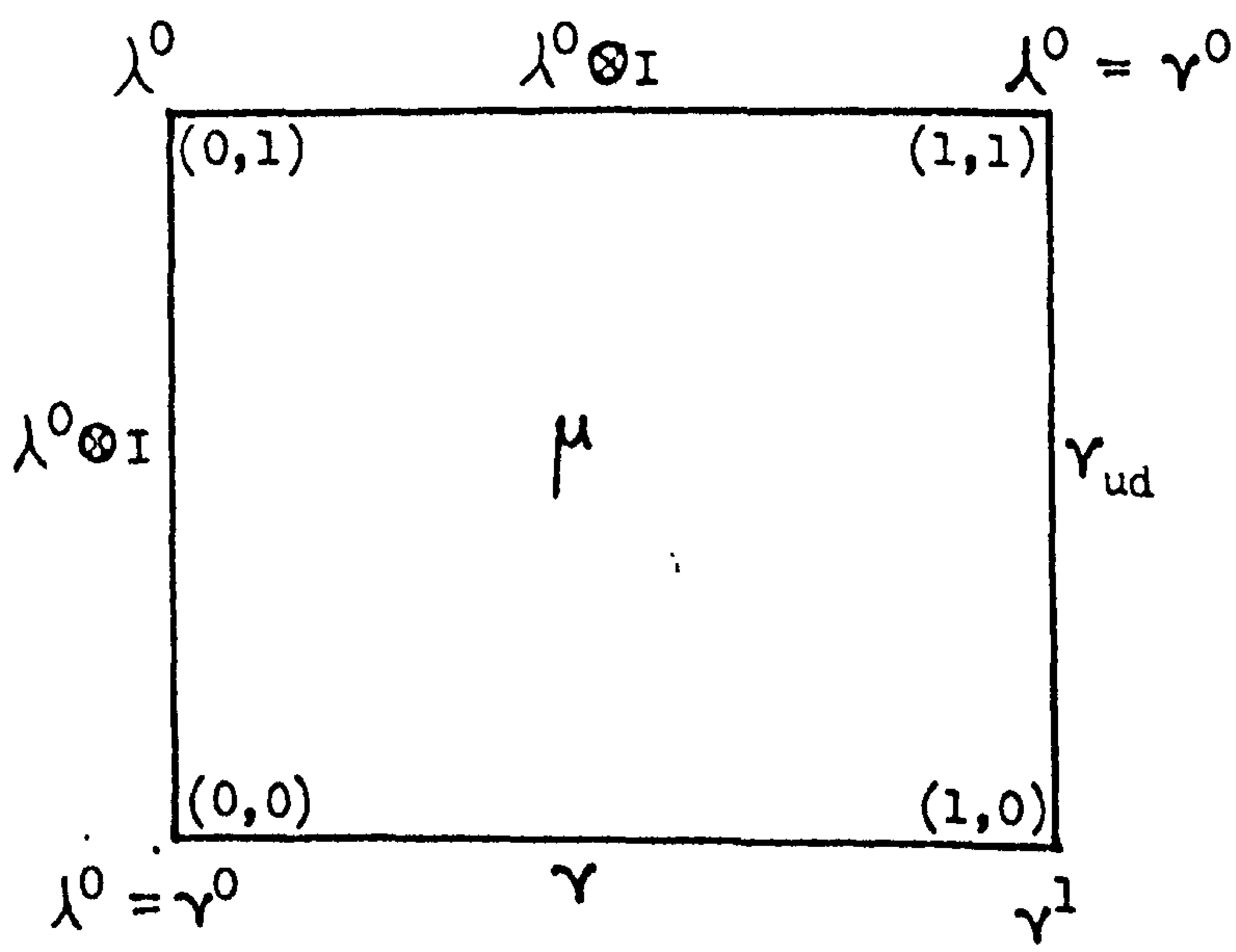
(i) x belongs to $F_k((C,\lambda^0)) \cap F_k((C,\lambda^1)) \subset F_k((C\otimes I,\lambda))$;

(ii) x belongs to $F_k((C,\gamma^0)) \cap F_k((C,\gamma^1)) \subset F_k((C\otimes I,\gamma))$.

Proof: W.l.o.g. , $\lambda = \lambda^0\otimes I$ (= pullback of λ^0 under the projection $C\otimes I \longrightarrow C$).

The proof consists in the definition of a suitable cycle $\mu \in Q(C\otimes I\otimes I)_0$ and an application of V.7 .

Since $I\otimes I$ is the cellular chain complex of the unit square $[0,1]\times[0,1]$, the following picture is meaningful and relevant.



It means that μ shall restrict

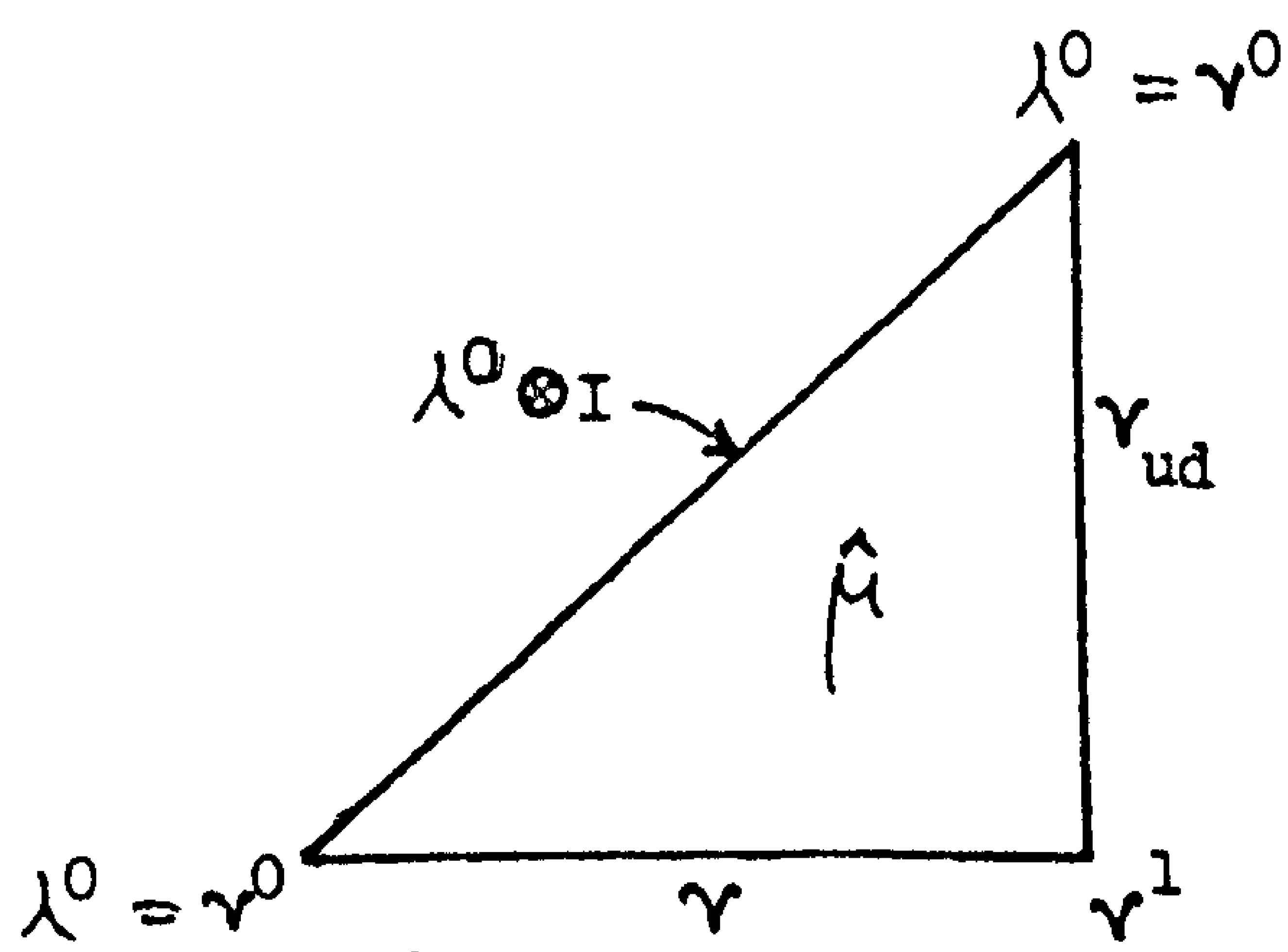
- to $\lambda^0\otimes I$ on $C\otimes\{0\}\otimes I \cong C\otimes I$, and on $C\otimes I\otimes\{1\} \cong C\otimes I$;

– to γ on $C \otimes I \otimes \{0\}$, and to γ_{ud} ($= \gamma$ upside down) on $C \otimes \{1\} \otimes I$.

(Such a cycle $\mu \in Q(C \otimes I \otimes I)_0$ can be found as follows: Start with

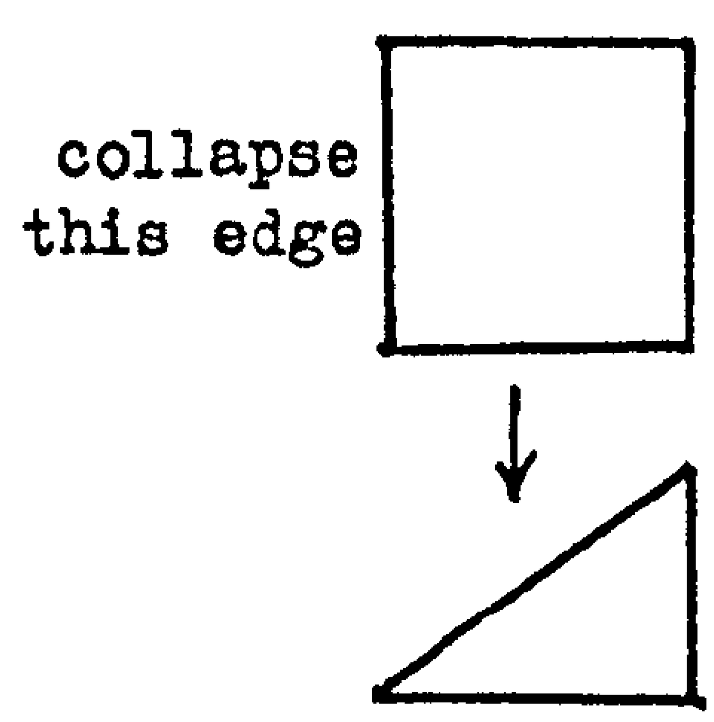
$$\gamma^0 \bullet\!\!\longrightarrow\!\!\bullet\, \gamma^1 \quad ,$$

considered as a 1-simplex in $Y(C)$; apply a suitable degeneracy operator to get a 2-simplex in $Y(C)$, i.e. a cycle $\hat{\mu} \in Q(C \otimes \Delta_2)_0$, as in the picture



;

use a collapsing map from the square to the triangle:



and let μ be the pullback of $\hat{\mu}$.)

Now for the proof: It suffices to prove only one half of the claim, say, the implication (i) $\Longrightarrow$ (ii) (the other half being similar); and it suffices to prove it with the group-valued functor F_k replaced by the monoid-valued functor $\mathcal{E}_k$ throughout.

Suppose then that $x \in \mathcal{E}_k((C,\lambda^0)) \cong \mathcal{E}_k((C\otimes\{(0,0)\},\lambda^0))$.
If (i) holds for x , there exists $x' \in \mathcal{E}_k((C\otimes\{(0,1)\},\lambda^0))$ so that x and x' map to the same element in $\mathcal{E}_k((C\otimes\{0\}\otimes I,\lambda^0\otimes I))$ or, for that matter, in $\mathcal{E}_k((C\otimes I\otimes I,\mu))$. Repeating this, one finds $x'' \in \mathcal{E}_k((C\otimes\{(1,1)\},\lambda^0))$ such that (I apologize for loose notation)

$$x'' = x' = x \quad \text{in} \quad \mathcal{E}_k((C\otimes I\otimes I,\mu)) .$$

Now the natural retraction from V.7 can be applied ,

$$r : \mathcal{E}_k((C\otimes I\otimes I,\mu)) \longrightarrow \mathcal{E}_k((C\otimes I\otimes\{0\},\gamma)) .$$

Since r is a retraction ,

$$r(x'') = r(x) = x \quad \text{in} \quad \mathcal{E}_k((C\otimes I\otimes\{0\},\gamma)) .$$

On the other hand, since r is natural , and since x'' belonged to $\mathcal{E}_k((C\otimes\{(1,1)\},\lambda^0)) \subset \mathcal{E}_k((C\otimes\{1\}\otimes I,\gamma_{ud}))$, $r(x'')$ belongs to $\mathcal{E}_k((C\otimes\{(1,0)\},\gamma^1))$.
Hence $x = r(x'') \in F_k((C\otimes\{(0,0)\},\gamma^0)) \cap F_k((C\otimes\{(1,0)\},\gamma^1))$, as required .

V.9 PROOF of II.4(a) and II.7 .
The functor F_k on $\tilde{\mathcal{C}}_\Lambda$ certainly has a subfunctor $\overline{F}_k^{pro}$ satisfying II.7 ; that is,

$$\overline{F}_k^{pro}((C,\lambda)) := \left\{ z \in F_k((C,\lambda)) \;\middle|\; i_{1*}(z) = i_{2*}(z) \right\} .$$

Proving II.4(a) and II.7 amounts to producing a factorization

$$\begin{tikzcd}
\tilde{\mathcal{C}}_\Lambda \arrow[r] \arrow[dr, "\overline{F}_k^{pro}"'] & \mathcal{H}\tilde{\mathcal{C}}_\Lambda \cong \mathcal{D}_0 \arrow[d, dotted] \\
 & \mathcal{G}roups
\end{tikzcd}$$
;

for such a factorization, if it exists, is unique (by definition of the quotient category) , and the dotted arrow then represents a functor which can be directly seen to have the universal property .

Suppose therefore that (C,λ^0) and (C',μ) are two objects in $\tilde{\mathcal{C}}_\Lambda$ (or $\mathcal{D}_0$) and that a morphism φ in $\mathcal{D}_0$ from (C,λ^0) to (C',μ) is given; as usual, this is represented by a pair $(f,[v])$, where $f:C \longrightarrow C'$ is a chain map and $[v]$ is a homotopy class of paths in $K(Q(C))$ joining the vertices λ^0 and $\lambda^1 := f^*(\mu)$.

OBSERVATION : The path class $[v]$ determines an isomorphism

$$j_{[v]} : \overline{F}_k^{pro}((C,\lambda^0)) \longrightarrow \overline{F}_k^{pro}((C,\lambda^1)) \quad .$$

(Proof: Represent $[v]$ by a 1-simplex in $K(Q(C))$, say v ; applying the realization procedure R to v (V.4) , obtain a cycle λ in $Q(C\otimes I)_0$ restricting to λ^0 and λ^1 on $C\otimes\{0\}$ and $C\otimes\{1\}$ respectively . Then, by V.8 ,

$$\overline{F}_k^{pro}((C,\lambda^0)) = F_k((C,\lambda^0)) \cap F_k((C,\lambda^1)) = \overline{F}_k^{pro}((C,\lambda^1)) \quad ,$$

all groups in this equation being regarded as subgroups of $F_k((C\otimes I,\lambda))$. This gives the identification; the choice of a 1-simplex to represent the class $[v]$ has no influence on the result .)

Hence, returning to the morphism $\varphi:(C,\lambda^0) \longrightarrow (C',\mu)$ in $\mathcal{D}_0$, we may define the induced homomorphism

$$\varphi_*: \bar{F}_k^{pro}((C,\lambda^0)) \longrightarrow \bar{F}_k^{pro}((C',\mu))$$

to be the composite

$$\bar{F}_k^{pro}((C,\lambda^0)) \xrightarrow[j_{[v]}]{} \bar{F}_k^{pro}((C,\lambda^1)) \xrightarrow[f_*]{} \bar{F}_k^{pro}((C',\mu))$$

(the chain map $f:C \longrightarrow C'$ is thought of as a morphism in $\tilde{\mathcal{C}}_\Lambda$, $f:(C,\lambda^1) \longrightarrow (C',\mu)$, and f_* stands for $\bar{F}_k^{pro}(f)$).

This definition of φ_* involves a representative $(f,[v])$ for the 'homotopy class' φ . If $(g,[w])$ is another representative for the same φ , there exists a third pair $(\Theta,[p])$, where

(i) $\Theta : C\otimes I \longrightarrow C'$ is a chain map and $[p]$ is a homotopy class of paths from $\lambda^0\otimes I$ to $\Theta^*(\mu)$;

(ii) $(\Theta,[p])$ 'restricts' to $(f,[v])$ on $C\otimes\{0\}$ and to $(g,[w])$ on $C\otimes\{1\}$.

Given that, it is easy to see (from the very definition of $\bar{F}_k^{pro}$) that φ_* is well defined . Checking functoriality is also a routine matter . -

V.10 PROOF of II.4(b) .

Given an object (C,λ^0) in $\tilde{\mathcal{C}}_\Lambda$, it must be shown that

$$\bar{F}_k^{pro}((C,\lambda^0)) = \left\{z \in F_k((C,\lambda^0)) \;\middle|\; i_{1*}(z) = i_{2*}(z)\right\}$$

is contained in the centre of $F_k((C,\lambda^0))$.

Let λ be a cycle in $Q(C\otimes I)_0$ restricting to λ^0 on $C\otimes\{0\}$, and so that the subcomplexes $C\otimes\{0\}$ and $C\otimes\{1\}$ of $C\otimes I$ are 'two-sided orthogonal' to each other, with respect to λ. That is, $\lambda_n(x,y) = 0 = \lambda_n(y,x)$ whenever $x \in (C\otimes\{0\})_n$, $y \in (C\otimes\{1\})_n$, for all n. (Proposition V.11 below shows how such a cycle λ can be obtained.)

Let $x \in \bar{F}_k((C,\lambda^0)) \cong \bar{F}_k((C\otimes\{0\},\lambda^0)) \subset F_k((C\otimes\{0\},\lambda^0))$.

Because of V.8, x then belongs to

$$F_k((C\otimes\{0\},\lambda^0)) \cap F_k((C\otimes\{1\},\lambda^1)) \qquad \subset \qquad F_k((C\otimes I,\lambda))$$

(λ^1 being the restriction of λ to $C\otimes\{1\}$).

But from the definition of the multiplication in $F_k((C\otimes I,\lambda))$, it is clear that the elements in the subgroup $F_k((C\otimes\{1\},\lambda^1))$ commute with those in the subgroup $F_k((C\otimes\{0\},\lambda^0))$, given that $C\otimes\{0\}$ is orthogonal to $C\otimes\{1\}$.

Hence x, being contained in the intersection of the two subgroups, is contained in the centre of each, q.e.d. .

V.11 PROPOSITION: Given two chain complexes C^0, C^1 in $\mathcal{C}_\Lambda$, the homomorphism

$$H_*(Q(C^0\oplus C^1)) \xrightarrow[(i_0^*,\ i_1^*)]{} H_*(Q(C^0)) \times H_*(Q(C^1))$$

(obtained by restricting to both summands) is an isomorphism.

Proof: Left to the reader.

The application to V.10 is as follows. Let λ^{pro} be <u>any</u> cycle in $Q(C\otimes I)_0$ extending λ^0 on $C\otimes\{0\}$; denote its restriction to $C\otimes\{1\}$ by λ^1.

Restricting λ^{pro} to $C\otimes\{0\}\oplus C\otimes\{1\} \subset C\otimes I$ gives a cycle in $Q(C\otimes\{0\}\oplus C\otimes\{1\})_0$; by V.11, this is homologous to the 'orthogonal sum' $\lambda^0 \oplus \lambda^1 \in Q(C\otimes\{0\}\oplus C\otimes\{1\})_0$.

Now the first of these cycles can be extended over $C\otimes I$, by construction; then, by an easy argument, the second also admits such an extension. But that is precisely what is needed in V.10 .

That finishes the proof sections. A few minor statements have actually not been proved yet, notably the assertions at the end of section III concerning the transformation $\circlearrowleft$. But they are all straightforward, as far as I can see .

VI. EXAMPLES AND APPLICATIONS

VI.1 Computation of the S-groups for 'framed bordism with double covers' .

Here $B := \mathbb{R}P^\infty$, and $\gamma : E \longrightarrow \mathbb{R}P^\infty = B$ is the trivial vector bundle.

By III.3 , the trivial bundle γ determines a homology class $v(\gamma)$ in $H_0(Q(C(\widetilde{\mathbb{R}P^\infty})))$; knowledge of this homology class implies knowledge of the groups $S_{2k}(\gamma)$ (provided the algebra is manageable) , though only up to non-unique isomorphism .

With the standard cell decomposition for $\mathbb{R}P^\infty$, $C(\widetilde{\mathbb{R}P^\infty})$ takes the form

$$\cdots - 0 \longleftarrow 0 \longleftarrow \mathbb{Z}[Z_2] \xleftarrow[\partial_1]{} \mathbb{Z}[Z_2] \xleftarrow[\partial_2]{} \mathbb{Z}[Z_2] \longleftarrow \cdots$$

with $\partial_n(1) = 1 + (-1)^n \tau$, τ being the generator of Z_2 .

VI.1(a) PROPOSITION: $v(\gamma) \in H_0(Q(C(\widetilde{\mathbb{R}P^\infty})))$ is represented by the cycle λ with

$$\lambda_n = \begin{cases} (1) & \text{if } n = 0 \\ (1+(-1)^n\tau) & n = 2^p \text{ for some } p \geq 0 \\ (0) & \text{otherwise} \end{cases}$$

for $n \geq 0$.

(Explanation: This is matrix notation; since each module $C(\widetilde{\mathbb{R}P^\infty})_n$ is identified with the free module on one generator, sesquilinear forms on it are identified with 1×1 -matrices with coefficient(s) in $\mathbb{Z}[Z_2]$.)

PROOF: Following the recipe in the proof of III.1, we shall construct a sequence of framed manifolds $(P_n^{2n})_{n=0,1,\dots}$ (each P_n mimicking the n-skeleton $\mathbb{R}P^n$ of $\mathbb{R}P^\infty$, etc.), and such that the sesquilinear sliding forms are as prescribed above.

Suppose we have already constructed $P_0, P_1, \dots, P_n$ so that the sliding forms are as prescribed (assume $n \geq 1$, otherwise there is little to prove).

CLAIM 1: Let z be a generator of $\pi_n(P_n) \cong \pi_n(\mathbb{R}P^n) \cong \pi_n(S^n)$. Because of Hirsch's immersion theorem, and because P_n is framed, z determines a regular homotopy class of immersions $i : S^n \looparrowright P_n$. The self-intersection number of this regular homotopy class, say $\mu(i)$, equals $1+\tau$ if $n = 2^p-1$ for some integer $p > 1$, and 0 otherwise. (It belongs to $\mathbb{Z}[Z_2]$ if n is even, to $\mathbb{Z}_2[Z_2]$ if n is odd.)

Assuming the truth of the claim, it is easy to construct P_{n+1}^{2n+2} etc. with the correct sliding form λ_{n+1}, thereby proving VI.1(a).

To prove the claim, we can suppose that n is odd. (In the even-dimensional case there is nothing to prove, because $\mu(i) \in \mathbb{Z}_2[Z_2]$ is algebraically determined by the sliding form λ_n: $2 \cdot \mu(i) = \lambda_n(1-\tau, 1-\tau)$, since i represents the element $1-\tau$ in $C(\widetilde{\mathbb{R}P^\infty})_n = \mathbb{Z}[Z_2]$.)

So let n be odd; let P_n' be a tubular neighbourhood of an embedded $\mathbb{R}P^n$ in $\mathbb{R}^{2n}$. Then there is a preferred homomotpy equivalence $P_n' \simeq \mathbb{R}P^n \simeq P_n$; since P_n' and P_n are both framed, we even have a preferred codimension zero immersion $e : P_n' \looparrowright P_n$.

Let $i' : S^n \looparrowright P_n'$ be the preferred immersion representing the generator of $\pi_n(P_n')$; thus $e \circ i'$ and i are regularly homotopic immersions of S^n in P_n. Now the self-intersection numbers $\mu(i')$ and $\mu(i)$

are easily seen to be equal, in spite of the self-intersections which the core of P'_n may have in P_n. (Every transverse self-intersection of this core gives a contribution of $\pm 2 \pm 2\tau = 0$ in $\mathbb{Z}_2[Z_2]$.)

Next, the self-intersection number $\mu(i')$ has the form $\alpha \cdot 1 + \beta \cdot \tau$ ($\alpha, \beta \in Z_2$) with $\alpha + \beta = 0$. For $\alpha + \beta$ is the self-intersection number of the composite immersion

$$S^n \xrightarrow[i']{} P'_n \longrightarrow \mathbb{R}^{2n} \quad ,$$

which by definition of i' is regularly homotopic to the standard embedding.

It remains to compute the coefficient α. This is the self-intersection number of the immersion $\tilde{i}'$ obtained by lifting i' to the double cover $\tilde{P}'_n$ of P'_n:

$$\begin{tikzcd} & \tilde{P}'_n \arrow[d] \\ S^n \arrow[ur, "\tilde{i}'"] \arrow[r, "i'"'] & P'_n \end{tikzcd} \quad .$$

Let δ be the self-intersection number (in $\mathbb{Z}_2$) of the core S^n of $\tilde{P}'_n$ in $\mathbb{R}^{2n}$ (since P'_n is immersed in $\mathbb{R}^{2n}$, so is its double cover; since P'_n is a tubular neighbourhood of an immersed $\mathbb{R}P^n$ in $\mathbb{R}^{2n}$, the double cover $\tilde{P}'_n$ is a tubular neighbourhood of an immersed S^n in $\mathbb{R}^{2n}$). Then clearly $\alpha + \delta = 0$. Hence all we need is

CLAIM 2: Let $j: \mathbb{R}P^n \looparrowright \mathbb{R}^{2n}$ be a smooth immersion, $p: S^n \longrightarrow \mathbb{R}P^n$ the standard twofold cover of $\mathbb{R}P^n$. Then the self-intersection number δ of the composite immersion $j \cdot p: S^n \looparrowright \mathbb{R}^{2n}$ (or of its regular homotopy class) is 1 if $n = 2^q - 1$ (for some integer $q \geq 1$), and 0 otherwise. (N.b.: δ lies in $\mathbb{Z}_2$ if n is odd, in $\mathbb{Z}$ otherwise.)

PROOF: For odd n, this is proved in [BROWN 2].
(Brown's computation is based on the following idea: Suppose there exists a bundle monomorphism $\eta \hookrightarrow \nu(j)$ over $\mathbb{R}P^n$, where η is the canonical line bundle, $\nu(j)$ the normal bundle of the immersion j. Then clearly $\flat = 0$. Bundle monomorphisms $\eta \hookrightarrow \nu(j)$ can be identified with sections of the bundle $\hom(\eta, \nu(j))$, which is isomorphic over $\mathbb{R}P^n$ to $\eta \otimes \nu(j)$. Hence it is plausible, and easy to verify, that the Euler number of $\eta \otimes \nu(j)$, which obstructs the existence of such a section, equals $\flat$ when n is odd, and the reduction of $\flat$ mod 2 when n is even.)
We do not need the even-dimensional case, but here is an outline: First, $\flat$ does not depend on the immersion j chosen. Second, $\flat = 0$ if and only if the normal bundle of the immersion $j \cdot p$ is trivial, if and only if the normal bundle $\nu(j)$ of the immersion j (a bundle over $\mathbb{R}P^n$) extends over $\mathbb{R}P^{n+1}$. But since $n+1$ is odd, there exists an immersion $\mathbb{R}P^{n+1} \looparrowright \mathbb{R}^{2n}$; if j is chosen to be the composite $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1} \looparrowright \mathbb{R}^{2n}$, then the normal bundle of j clearly extends over $\mathbb{R}P^{n+1}$.

This completes the proof of VI.1(a).

For the corollary below, note that there is a transfer homomorphism (see II.6 etc.)

$$S_{2k}(\gamma) \longrightarrow S_{2k}(\tilde{\gamma}) \cong S_{2k}(*)$$

(where $\tilde{\gamma}$ is a certain bundle over the double cover S^∞ of $\mathbb{R}P^\infty$, and $*$ stands for the trivial bundle over a point).

VI.1(b) COROLLARY: For odd k , the transfer

$$S_{2k}(\gamma) \longrightarrow S_{2k}(*) \cong Z_2$$

is surjective if and only if $k = 2^p-1$ for some integer $p \geq 1$.

PROOF. The group ring $\Lambda = \mathbb{Z}[Z_2]$ has two special properties: (i), any f.g. stably free module over Λ is free; (ii) , if $\lambda' : E \times E \longrightarrow \Lambda$ is a (-1)-hermitian form on the f.g. free module E , then $\lambda'(x,x) = 0$ for every $x \in E$.

Consequently, every simple nondegenerate (-1)-hermitian form $\lambda' : E \times E \longrightarrow \Lambda$ splits as an orthogonal sum of a finite number of 'standard hyperbolic planes' , each having rank two (i.e. being defined on a free module on two generators). This fact can be exploited for the analysis of $F_k((C(\widetilde{\mathbb{R}P^\infty}),\lambda))$ (with λ as in VI.1(a)). It shows that $F_k((C(\widetilde{\mathbb{R}P^\infty}),\lambda))$ is generated by 'extensions of rank two' , that is, by extensions (D,i,f,b,q) such that $D/_{im(i)}$ is free on two generators (cf. section IV).

It is then easy to check by hand that the transfer maps each of these rank two generators to 0 , provided $k \neq 2^p-1$.

The case $k = 2^p-1$ can also be done by hand (n.b.: this requires determination of $\bar{F}_k((C(\widetilde{\mathbb{R}P^\infty}),\lambda)) \subset F_k((C(\widetilde{\mathbb{R}P^\infty}),\lambda))$) , but alternatively the existence of framed manifolds of Kervaire invariant 1 in low dimensions can be used together with the argument employed to prove

VI.1(c) COROLLARY (Browder's theorem):

If $k+1$ is not a power of 2 , the Kervaire invariant

$$\Phi : \pi^s_{2k} \longrightarrow Z_2$$

is trivial. (Proof: see introduction .)

VI.2 S-theory with coefficients Z_2 .

As stated in II.3(iv) and II.4(b) , any homomorphism of rings with involution $\varphi : \Lambda \longrightarrow \Lambda'$ induces a functor

$$- \otimes_\Lambda \Lambda' : \mathcal{H}\tilde{\mathcal{C}}_\Lambda \longrightarrow \mathcal{H}\tilde{\mathcal{C}}_{\Lambda'}$$

and a transformation of functors from $\bar{F}_{k,\Lambda}$ to $\bar{F}_{k,\Lambda'} \cdot - \otimes_\Lambda \Lambda'$.
In particular, if $\gamma : E \longrightarrow B$ is a stable vector bundle over the CW-space B (with finitely many cells in each dimension, say) , and Λ is the group ring of $\pi_1(B)$, with the usual involution , then γ determines an object in $\mathcal{H}\tilde{\mathcal{C}}_\Lambda$ (III.1 or III.7) ;
the functor $\bar{F}_{k,\Lambda'} \cdot - \otimes_\Lambda \Lambda'$ can be applied to that object and yields an " S-group with coefficients Λ' " ,

$$S_{2k}(\gamma ; \Lambda') \quad \text{or perhaps} \quad S_{2k}(\gamma ; \varphi) \quad ,$$

the latter if it is not clear which homomorphism φ is meant.
The transformation just mentioned gives natural homomorphisms

$$S_{2k}(\gamma) \longrightarrow S_{2k}(\gamma ; \Lambda')$$

(though natural only 'where applicable').

It turns out that ' S-theory with coefficients Z_2 ' is nothing but an old friend. First of all, ' toy bundles with coefficients Z_2 ' and the like are easy to understand because of

VI.2(a) PROPOSITION: For a chain complex C in $\mathcal{C}_\Lambda$,

$$H_0(Q(C \otimes_\Lambda Z_2)) \cong \prod_n H^n(C;Z_2) \cong H_1(Q(C \otimes_\Lambda Z_2)) \quad .$$

(Cf. II.1 . The isomorphisms are canonical; it is understood that Λ is a group ring as above, and that Q is a functor on $\mathcal{C}_{Z_2}$ (II.2) .)

PROOF: Observe first that $Q(C\otimes_\Lambda Z_2)$ is periodic with period one, hence only one of the two isomorphisms need be established.

Next, $H_0(Q(\ldots))$ is a homotopy invariant functor (III.4) , and $C\otimes_\Lambda Z_2$ is canonically homotopy equivalent (as a chain complex) to its homology $H_*(C;Z_2)$, considered as a chain complex with zero differential. Therefore $H_0(Q(C\otimes_\Lambda Z_2)) \cong H_0(Q(H_*(C;Z_2)))$.

Now a cycle λ in $Q(H_*(C;Z_2))$ is just a sequence of symmetric bilinear forms

$$\lambda_n : H_n(C;Z_2)\times H_n(C;Z_2) \longrightarrow Z_2 \quad ;$$

the cycle is a boundary precisely if each λ_n has a decomposition

$$\lambda_n = \mu_n + T(\mu_n)$$

for some bilinear form $\mu_n : H_n(C;Z_2)\times H_n(C;Z_2) \longrightarrow Z_2$.

But such a decomposition for λ_n exists if and only if the homomorphism

$$\begin{aligned} sq(\lambda_n) : H_n(C;Z_2) &\longrightarrow Z_2 \\ z &\longmapsto \lambda_n(z,z) \end{aligned}$$

is zero . In other words, the required isomorphism is obtained by associating with every class $[\lambda]$ in $H_0(Q(H_*(C;Z_2)))$ the sequence of elements $sq(\lambda_n)$ in $H^n(C;Z_2)$.

'Tensoring' III.3 with Z_2 and combining with the preceding result gives

VI.2(b) OBSERVATION: Every real stable vector bundle $\gamma : E \longrightarrow B$ determines a class $v(\gamma;Z_2)$ in $H_0(Q(C(\tilde{B})\otimes_\Lambda Z_2) \cong \prod_n H^n(B;Z_2)$.

VI.2(c) PROPOSITION: $v(\gamma;Z_2)$ is the total Wu class of γ .

Proof: Write $\Lambda = \pi_1(B)$ as usual. Choose a γ-fattening $(P_n^{2n})_{n=0,1,\ldots}$ of B (see proof of III.1), and let $(C(\tilde{B}),\lambda)$ be the associated object in $\tilde{\mathcal{E}}_\Lambda$ (III.1) .

We are interested in $(C(\tilde{B}),\lambda)\otimes_\Lambda Z_2 = (C(\tilde{B})\otimes_\Lambda Z_2\,,\lambda\otimes_\Lambda Z_2)$.

Now it is important to realize that instead of constructing the cycle λ as in the proof of III.1 and then tensoring with Z_2 (over Λ) , we can rewrite III.1 'with coefficients Z_2 ' .

Thus $\quad (C(\tilde{B})\otimes_\Lambda Z_2)_n = C(\tilde{B})_n\otimes_\Lambda Z_2 = H_n(P_n,P_{n-1}\times I;\Lambda)\otimes_\Lambda Z_2$

$$\cong H_n(P_n,P_{n-1}\times I;Z_2)\quad ,$$

and similarly the bilinear form $\lambda_n\otimes_\Lambda Z_2$ can be reinterpreted or defined directly as a sliding form on $H_n(P_n,P_{n-1}\times I;Z_2)$.

We are really only concerned with the kernel of

$$\partial_n : H_n(P_n,P_{n-1}\times I;Z_2) \longrightarrow H_{n-1}(P_{n-1},P_{n-2}\times I;Z_2) \quad ;$$

given $x \in \ker(\partial_n)$, we must show that

$$(*)\qquad (\lambda_n\otimes_\Lambda Z_2)(x,x) = v_n(\gamma)\cdot[x]$$

where $v_n(\gamma)$ is the n-th Wu class of γ in $H^n(B;Z_2)$, and $[x] \in \ker(\partial_n)/\mathrm{im}(\partial_{n+1}) \cong H_n(B;Z_2)$ the homology class represented by x .

But $\quad \ker(\partial_n) = \mathrm{im}\left[H_n(P_n;Z_2) \longrightarrow H_n(P_n,P_{n-1}\times I;Z_2)\right]$,

and if $\hat{x} \in H_n(P_n;Z_2)$ maps to $x \in \ker(\partial_n)$, then we recognize $\lambda_n\otimes_\Lambda Z_2(x,x)$ as the homology intersection product of x with itself.

Further, we may represent x by a map from a closed manifold N^n to P_n^{2n} , say $f:N^n \longrightarrow P_n$. Then $\lambda_n\otimes_\Lambda Z_2(x,x)$

$$= \text{Euler number of the normal bundle of } N^n \text{ in } P_n^{2n}$$

$= w_n(\,\gamma_N - f^*i^*(\gamma)) \cdot [N^n]$

(w_n = n-th Stiefel-Whitney class; γ_N = normal bundle of N ;
$i : P_n \longrightarrow B$ is the composite $P_n \simeq$ n-skeleton of $B \hookrightarrow B$)

$= v_n(f^*i^*(\gamma)) \cdot [N^n]$ (by what appears to be a well-known formula)

$= v_n(\gamma) \cdot i_*f_*([N]) = v_n(\gamma) \cdot [x]$, which proves ($*$) .

The preceding results can be used to show that 'S-theory with coefficients Z_2 ' is more or less identical with the theory in [BROWN 1] . This claim needs some explanation.
For certain Thom spectra M , [BROWN 1] constructs Kervaire-Arf invariants $\Phi : \pi_{2k}(M) \longrightarrow Z_8$; his construction uses a non-canonical homomorphism $h : M_{2k}(K(Z_2,k)) \longrightarrow Z_4$ at some stage, $K(Z_2,k)$ being an Eilenberg-MacLane space . (Admittedly, the theory of [BROWN 1] is more general than that .) However, it is rather clear that one can do without any such homomorphisms, at the cost of letting the Kervaire invariant have values in a slightly more complicated Witt-Grothendieck group $W_{2k}(M)$.
Each of the (admissible) homomorphisms h referred to above gives an isomorphism

$$W_{2k}(M) \cong Z_8 \oplus \ker\left[Sq_k : H_k(M;Z_2) \longrightarrow H_0(M;Z_2) \cong Z_2\right]$$

or $$W_{2k}(M) \cong Z_2 \oplus H_k(M;Z_2) \quad ,$$

depending on whether or not $Sq_k : H_k(M;Z_2) \longrightarrow H_0(M;Z_2) \cong Z_2$ is onto.
(Warning: See [BROWN 1] or [SWITZER] for the definition of Sq_k as a homology operation.)

VI.2(d) PROPOSITION: $W_{2k}(M)$ is naturally isomorphic to $S_{2k}(\gamma;Z_2)$, for a Thom spectrum $M = M(\gamma)$.

(Proof:omitted, but not difficult . The proposition makes sense only if $M(\gamma)$ satisfies Brown's conditions; in the present case, this means that the Wu class $v_{k+1}(\gamma)$ in $H^{k+1}(B;Z_2)$ is zero, B being the base space of γ .)

VI.3 The Brown-Gitler spectrum and the Kervaire invariant problem.

Here we follow the plot outlined in the introduction (section 0); this means that, first of all, a description of the Brown-Gitler spectrum as a Thom spectrum is needed .

VI.3(a) EXAMPLE ([BROWN-GITLER] ; [BROWN-PETERSON 1,2] ; work done by Ralph Cohen , for which I have no references) :
Let BBr_{2k} be the space of subsets of $\mathbb{R}^2 = \mathbb{C}$ having 2k elements (this is a $K(\pi,1)$, and its fundamental group is called the 'braid group on 2k strings'); let $\gamma : E \longrightarrow BBr_{2k}$ be the 2k-dimensional vector bundle over BBr_{2k} whose fibre at $\{x_1,x_2,\dots,x_{2k}\} \in BBr_{2k}$ is the vector space of functions $\{x_1,x_2,\dots,x_{2k}\} \longrightarrow \mathbb{R}^1$.
Then the Thom space of γ can be regarded as a Thom spectrum, and this is known to satisfy all the conditions which characterize the Brown-Gitler spectrum $B(k)$.
Sometimes it helps to know that BBr_{2k} is a framed manifold (because it can be identified with the space of complex polynomials of degree 2k , with leading coefficient 1 and no multiple roots) ; and that $2\cdot\gamma$ has a canonical trivialization, being practically identical with the tangent bundle of BBr_{2k} ([FUKS]) .

The aim is now to find useful conditions which an element x in $\pi_{2k}(B(k))$ must satisfy if it wants to belong to the image of $g_* : \pi^s_{2k} \longrightarrow \pi_{2k}(B(k))$ (g_* is induced by the generator $g : S^0 \longrightarrow B(k)$ of $\pi_0(B(k)) \cong Z_2$).

The following two lemmas (the first is just a weaker version of the second, with a more illuminating proof) lead to such a condition. Apparently it is only satisfied by a single element (apart from 0) in $\pi_{2k}(B(k))$ (for $k = 2^p-1$), which would be ideal.

VI.3(b) LEMMA: Suppose $z \in \pi_{2k}^s \cong \pi_{2k}^{*)}(\Omega^\infty\Sigma^\infty S^0)$ belongs to the kernel of the Hurewicz homomorphism

$$\pi_{2k}(\Omega^\infty\Sigma^\infty S^0) \longrightarrow H_{2k}(\Omega^\infty\Sigma^\infty S^0;Z_2) .$$

Then z also belongs to the kernel of $\pi_{2k}^s \longrightarrow \pi_{2k}(B(k))$.

Proof: Write Y for $\Omega^\infty\Sigma^\infty S^0$.

If z belongs to the kernel of $\pi_{2k}(Y) \longrightarrow H_{2k}(Y)$ (coefficients Z_2), then also to the kernel of $\pi_{2k}(Y) \longrightarrow \eta_{2k}(Y)$; η denotes unoriented bordism.

Thus if $f:S^{2k} \longrightarrow Y$ represents z, we can find a smooth compact manifold M^{2k+1} with $\partial M = S^{2k}$, and an extension

$$\bar{f}: M \longrightarrow Y \qquad (\text{so } \bar{f}/_{S^{2k}} = f) .$$

Now $M \bigcup_{S^{2k}} D^{2k+1}$ is a closed manifold of dimension $2k+1$ (D^{2k+1} = disk); therefore a map $g: N^{2k+1} \longrightarrow M \cup D^{2k+1}$ of Z_2-degree 1 exists, where N is an MBr_{2k}-manifold .

(Explanation: $MBr_{2k} = M(\gamma) \simeq B(k)$ is the Thom spectrum from VI.3(a) . The claim follows from one of the defining properties of the Brown-Gitler spectrum .)

We may suppose g to be locally diffeomorphic in a neighbourhood of $g^{-1}(D)$, so that $g^{-1}(D)$ is an odd number of copies of D .

*) This is an unstable homotopy group .

Then, looking at $g^{-1}(M)$, we find that $p\cdot z$ belongs to the kernel of β (where $\beta : \pi_{2k}(Y) \longrightarrow (MBr_{2k})_{2k}(Y)$ is the forgetful homomorphism, which considers a sphere as a framed, and hence braided, manifold); p is the odd number mentioned earlier.
But we also have $2\cdot\beta \equiv 0$ (by definition of β ; note that $\pi_0(MBr_{2k}) \cong Z_2$). So $\beta(z) = 0$, and the remainder of the proof is easy.

(Lemma VI.3(b) is related to the following popular conjecture:

VI.3(c) CONJECTURE: $z \in \pi_i^s$ belongs to the kernel of the Hurewicz homomorphism $\pi_i^s \cong \pi_i(\Omega^\infty\Sigma^\infty S^0) \longrightarrow H_i(\Omega^\infty\Sigma^\infty S^0;Z_2)$ if and only if it has

- Hopf invariant 0 (for i odd)
- Kervaire invariant 0 (for i even) .

This conjecture has some sort of history; Curtis claimed to have a proof (E.B. CURTIS , The Dyer-Lashof algebra and the Λ-algebra , Illinois Journal of Mathematics 19 (1975) , 231 - 246) , but J.P.May apparently spotted a gap in it, and one of May's students has written a thesis about the problem .)

VI.3(d) LEMMA: Let Y be any connected CW-space, and let $P(H_{2k}(Y;Z_2)) \subset H_{2k}(Y;Z_2)$ be the subgroup consisting of the primitive elements which are annihilated by all Steenrod squares. There is a natural homomorphism $\Psi : P(H_{2k}(Y;Z_2)) \longrightarrow \widetilde{B(k)}_{2k}(Y) = B(k)_{2k}(Y,*)$ so that the following diagram is commutative:

$$\begin{tikzcd}
\pi_{2k}(Y,*) \arrow[d] \arrow[dr] & \\
P(H_{2k}(Y,*;Z_2)) \arrow[r, "\Psi"] & B(k)_{2k}(Y,*)
\end{tikzcd}$$

and so that $? \cdot \Psi : P(H_{2k}(Y,*;Z_2)) \longrightarrow H_{2k}(Y,*;Z_2)$ is the inclusion ($? : B(k) \longrightarrow HZ_2$ being the generator of $H^0(B(k);Z_2) \cong Z_2$).

To describe Ψ, we have to use some of the results stated in [BROWN-GITLER]. Brown and Gitler construct $B(k)$ in a roundabout way, namely by first constructing a spectrum $\chi(B(k))$, the 'Pontryagin dual' of $B(k)$. This is related to $B(k)$ by Pontryagin duality, as one would expect; that is, for any finite based CW-space Y and any integer i, $B(k)_i(Y,*)$ and $\chi(B(k))^i(Y,*)$ are finite groups, and

$$B(k)_i(Y,*) = \left[\chi(B(k))^i(Y,*)\right]^t := \mathrm{Hom}\left[\chi(B(k))^i(Y,*)\ ,\ S^1\right] .$$

(Of course, induced homomorphisms etc. are also related by duality.) Thus, for instance, $\pi_i(B(k)) = [\pi_{-i}(\chi(B(k))]^t$.

The properties of $\chi(B(k))$ (as stated in [BROWN-GITLER, 5.1 (iv)]) imply rather easily that $\chi(B(k))_{2k}$ (= CW-space representing the functor $\chi(B(k))^{2k}(-)$) is equivalent, though not by a well-defined homotopy equivalence, to a certain product of Eilenberg-Mac Lane spaces $K(Z_2,i)$. This is also mentioned explicitly in [BROWN-PETERSON 2]. The equivalence is one of CW-spaces, but not of infinite loop spaces; thus

$$(*) \qquad \pi_i(\chi(B(k))_{2k}) \cong \pi_i\Big(\prod_{j\in I} K(Z_2,i_j)\Big)$$

where the isomorphism is a group isomorphism if $i>0$, but only a set isomorphism if $i=0$ (although even then, both π_0's are abelian groups) . In fact, it is shown in [BROWN-PETERSON 2] that $\pi_0(\chi(B(k))_{2k}) \cong [\pi_{2k}(B(k))]^t$ usually contains elements of order higher than two. Anyway, the homotopy groups $\pi_i(\chi(B(k))_{2k})$ are determined by $(*)$, at least for $i>0$; in particular we have, for $i<k$,

$$(**) \qquad \pi_{2k-i}(\chi(B(k))_{2k}) \cong \pi_i(B(k)) \cong \begin{cases} 0 & \text{if} \quad i<0 \\ Z_2 & \text{if} \quad i=0 \\ 0 & \text{if} \quad 0<i<k \end{cases}$$

which can also be deduced directly by looking at $H^*(B(k);Z_2)$.

Now let $x \in P(H_{2k}(Y;Z_2))$; it suffices to consider the case where Y is a compact CW-space . We shall construct a homomorphism $e_x : \chi(B(k))_{2k}(Y,*) \longrightarrow Z_2$ as follows. Consider the composite

$$\begin{tikzcd}
{[Y,\ \chi(B(k))_{2k}]} \arrow[r, "(1)"'] & {[Y,\ \prod_{j\in I} K(Z_2\ ,\ i_j)]} \arrow[d, "(2)"] & \\
& {[Y,\ K(Z_2,2k)]} \arrow[r, "(3)"'] & Z_2
\end{tikzcd}$$

where square brackets denote morphism sets, arrow (1) comes from $(*)$, (2) is induced by the projection onto the top-dimensional factor (see $(**)$) , and (3) is the scalar product with x .

Arrow (1) is neither well defined nor a group isomorphism. Claim: the composite $e_x := (3)\cdot(2)\cdot(1)$ is nevertheless well defined and a group homomorphism.(Proof: Exercise.) Dualizing e_x , we obtain

$$e_x^t : Z_2 \longrightarrow [(B(k))^{2k}(Y,*)]^t \cong B(k)_{2k}(Y,*) \quad ,$$

and put $\psi(x) := e_x^t(1)$.

VI.3(e) OBSERVATION: The diagram

$$\begin{array}{ccc} \pi^s_{2k} & \longrightarrow & P(H_{2k}(\Omega^\infty\Sigma^\infty S^0);Z_2) \\ \downarrow & & \downarrow \psi \\ \pi_{2k}(B(k)) & \xleftarrow{\varphi} & \widetilde{B(k)}_{2k}(\Omega^\infty\Sigma^\infty S^0) \end{array}$$

is commutative.

(Explanation: $\varphi\colon \widetilde{B(k)}_{2k}(\Omega^\infty\Sigma^\infty S^0) \longrightarrow \pi_{2k}(B(k)) \cong \widetilde{B(k)}_{2k}(S^0)$ is induced by the usual collapse $r\colon \Omega^\infty\Sigma^\infty S^0 \longrightarrow S^0$, a stable map. Since $B(k) \simeq MBr_{2k}$ is a Thom spectrum, φ can also be described in the following manner (in any dimension n). Regard $\Omega^\infty\Sigma^\infty S^0$ as the representing space for framed cobordism of codimension zero. Any $x \in (MBr_{2k})_n(\Omega^\infty\Sigma^\infty S^0)$ is represented by some map $N^n \longrightarrow \Omega^\infty\Sigma^\infty S^0$, where N is a closed nulbordant MBr_{2k}-manifold ; this gives an element in $\eta^0_{fr}(N)$, represented in turn by a map $N' \longrightarrow N$. Then $\varphi(x) := [N']$.

According to conjecture VI.3(c), the image of $\pi^s_{2k} \longrightarrow P(H_{2k}(\Omega^\infty\Sigma^\infty S^0;Z_2)$ has order two at most. Moreover, in his Illinois paper, E.B.Curtis describes a single nonzero candidate (for being spherical) in $P(H_{2k}(\Omega^\infty\Sigma^\infty S^0;Z_2)$, provided $k = 2^p-1$.
By VI.3(e), this determines another 'candidate' in $\pi_{2k}(B(k))$, say z, and so the question

" Is $\partial(z)$ equal to $j(1)$? "

(cf. introduction) seems rather tantalizing, if remote. The answer may, of course, depend on which Thom spectrum is chosen to represent $B(k)$.

REFERENCES:

BROWDER: The Kervaire invariant of framed manifolds and its generalisations, Ann. of. Math. 90(2), 1969, 157 - 186 .

BROWN 1: Generalisations of the Kervaire invariant, Ann. of Math. 95(2), 1972, 368 - 383 .

BROWN 2: A remark concerning immersions of S^n in $\mathbb{R}^{2n}$, Quart.J.Math.Oxford(2), 24, 1973, 559 - 560 .

BROWN-GITLER: A spectrum whose cohomology is a certain cyclic module over the Steenrod algebra, Topology 12, 1973, 283 - 295 .

BROWN-PETERSON 1: On the stable decomposition of $\Omega^2 S^{r+2}$, Trans.Amer.Math.Soc. 243, 1978, 287 - 298 .

BROWN-PETERSON 2: The Brown-Gitler spectrum, $\Omega^2 S^3$, and $\eta_j \in \pi_{2^j}(S)$, Preprint (has probably appeared somewhere) , Brandeis University, M.I.T. .

CURTIS : Simplicial homotopy theory, Advances in Math. 6(2), 1971, 107 - 209 .

DOLD : Lectures on Algebraic Topology , Vol.200 of Grundlehren, Springer Verlag Berlin - Heidelberg - New York , 1972 .

FUKS : Cohomology of the braid group, Funktsionalnyi Analyz i Ego Prilozheniya, 4(2) , 1970 , 62 - 73 = Functional Anal. Appl. 4 , 1970, 143 - 151 .

GABRIEL-ZISMAN : Calculus of Fractions and Homotopy Theory , Vol.35 of Ergebnisse, Springer Verlag Berlin-Heidelberg-NewYork,1967.

JONES-REES: Kervaire's invariant for framed manifolds, Amer.Math.Soc.: Proceedings of Symposia ... , 32(1) , 1978 , 141 - 147 .

KAHN-PRIDDY : Applications of the transfer to stable homotopy theory, Amer.Math.Soc.Bulletin 78, 1972, 981 - 987 .

MILNOR 1: Lectures on the h-cobordism theorem, Mathematical Notes, Princeton University Press, 1965 .

MILNOR 2: Whitehead Torsion, Amer.Math.Soc.Bulletin 72, 1966, 358 - 426 .

QUINN : Open book decompositions, and the bordism of automorphisms, Topology 18, 1979, 55-73 .

STONG: Notes on Cobordism Theory, Mathematical notes,
Princeton University Press, 1958 .

SWITZER: Algebraic Topology - Homotopy and Homology, Vol.212 of Grundlehren,
Springer Verlag Berlin-Heidelberg-New York, 1975 .

WALL 1: Surgery on compact manifolds, Academic press London-New York, 1970 .

WALL 2: On the axiomatic foundations of the theory of hermitian forms,
Proc.Camb.Phil.Soc. 67, 1970, 243 - 250 .

WALL 3: "Differential Topology" , Notes, Liverpool University, 1965,
Part IV: Theory of handle decompositions .